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THESIS

OPTIMIZATION OF ARTIFICIAL DISPERSION IN SALVO FIRING

by

Takasi Kisi

December 1983

Thesis Advisor: A. R. Washburn

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T215229

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Optimization of Artificial Dispersion in Salvo Firing		5. TYPE OF REPORT & PERIOD COVERED Ph.D. Thesis; December 1983
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Takasi Kisi		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93943		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, California 93943		12. REPORT DATE December 1983
		13. NUMBER OF PAGES 117
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Firing
Coverage problem
Salvo kill probability
Optimal ballistic dispersion

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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#20 - ABSTRACT - (CONTINUED)

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Optimization of Artificial Dispersion in Salvo Firing

by

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Submitted in partial fulfillment of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

from the

NAVAL POSTGRADUATE SCHOOL
December 1983

CALIFORNIA 939

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In salvo firing, the smallest ballistic dispersion is not always most desirable. Deliberate increase of the ballistic dispersion can improve the probability of destroying the target. Our concern in this dissertation is the optimization of such "artificial" dispersion in two-dimensional salvo models. In some cases no closed form solution is available, but we are able to offer efficient methods for the computation or approximation of the salvo kill probability. In other cases we are able to derive approximate formulae for the optimal ballistic dispersion.

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I. INTRODUCTION

Let us begin with a classical example presented in Chapter 6 of the textbook by Morse and Kimball [Ref. 1]. Suppose an airplane carries two bombs to attack a railroad track. The plane flies along a course perpendicular to the track, and drops the bombs. There are three ways to drop two bombs: (1) together in salvo, (2) spaced a suitable distance apart aiming the midpoint of the pattern at the center of the track, and (3) dropping each on a separate run over the target.

We intend to illustrate the method of calculating the probability of track destruction in the three cases, the purpose being to determine which method of attack is the best.

Let us consider Case (1), salvo bombing. The two bombs leaving the plane simultaneously will hit the ground at some distance apart; the impact point being random. For the time being the random vector of the impact position from the center of impact is termed the ballistic error. The aiming is also not free from error. The plane aims at a point on the center of the track, but the center of impact will deviate from the aim point. This deviation, or aiming error, is also random. The impact point of each bomb is, therefore, composed of two errors; first the aiming error which

is common to both bombs, and second the ballistic error which varies from one bomb to another.

If the variance of the ballistic error is small while that of the aiming error is large, the two bombs will land almost at the same point. Therefore it is expected that either both will hit or neither will hit the track. To avoid the latter situation and to improve the probability of destruction, it is better to spread the landing pattern of the bombs. One way is the method of pattern bombing described as Case (2), another is salvo bombing employing bombs with a suitable value of variance of the ballistic error. Our main concern in this dissertation is maximization of the probability of salvo destruction by choosing a suitable value of the variance of the ballistic error. In the following we call it the optimal ballistic dispersion.

The problem described above is not a new one. In an interesting talk before a conference at the Ballistic Research Laboratories held in March 1955 [Ref. 2], Merritt and King stated that an approximate formula for the optimal ballistic dispersion was derived by two Englishmen as early as in 1936. Following the work in WW II, a number of articles were published on both the salvo and pattern firing models, but since the early 1970's, it seems that these models have attracted less attention. Further detail can be found in two excellent review papers by Eckler [Ref. 3] and Eckler and Burr [Ref. 4]. However, little is known as to the

optimization of the ballistic dispersion in salvo models. It is the calculation or approximation of the optimal ballistic dispersion that is the subject of this dissertation.

A. SALVO FIRING

In the following chapters we deal with only two-dimensional salvo models. The reason we adopt the two-dimensional model is that it is the most frequent case and plays an important role in real world applications. One- or three-dimensional theory can also be developed to parallel our investigation.

Suppose that there is a target in a two-dimensional space, and a salvo of n weapons is delivered against the target. Delivery error relative to the target is assumed to be composed of two parts, the aiming error and the ballistic error.

First we aim at the target. With this aiming, the center of impact point of n weapons is determined. Let us adopt a Cartesian coordinate system (x,y) such that the origin coincides with the center of impact. The aiming cannot be perfect. Let the components of the aiming error be $-U$ and $-V$. Then the position of the target with respect to our coordinate system is given by (U,V). It is assumed that the joint probability density of U and V exists and is given by $f_1(u,v)$.

Now, a salvo of n weapons is fired against the target after (U,V) takes a value (u,v) which is unknown to us and only is predictable in a probabilistic sense. The components of the impact point of the ith weapon are denoted as X_i and Y_i. It is assumed that (X_i,Y_i) are independent and identically

distributed random variables with joint density $f_2(x,y)$. In Fig. 1, a typical geometry is shown. The target is at (u,v), and the impact points of four weapons are scattered around the origin.

As to the target we adopt the so-called point target concept: It is assumed that the target is either completely destroyed or else undamaged by each weapon. We neglect any possible partial damage and its cumulative effect. The probability that a weapon landing at (x,y) destroys the target at (u,v) is a function of u-x and v-y, denoted as D(u-x,v-y) and is called the damage function.

The probability of destroying the target is calculated in two steps. First, suppose that the random variables U and V take some values u and v. The conditional probability that the ith weapon destroys the target given U = v and V = v is

$$p(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(u-x,v-y) f_2(x,y)\, dxdy \; . \qquad (1.1)$$

The impact points X_i and Y_i are all assumed to be independent, so the conditional survival probability of the target given U = u and V = v is $\{1 - p(u,v)\}^n$. Therefore, we obtain the probability that the target is destroyed by the salvo of n weapons--salvo kill probability--by averaging the conditional survival probability with respect to the distribution of u

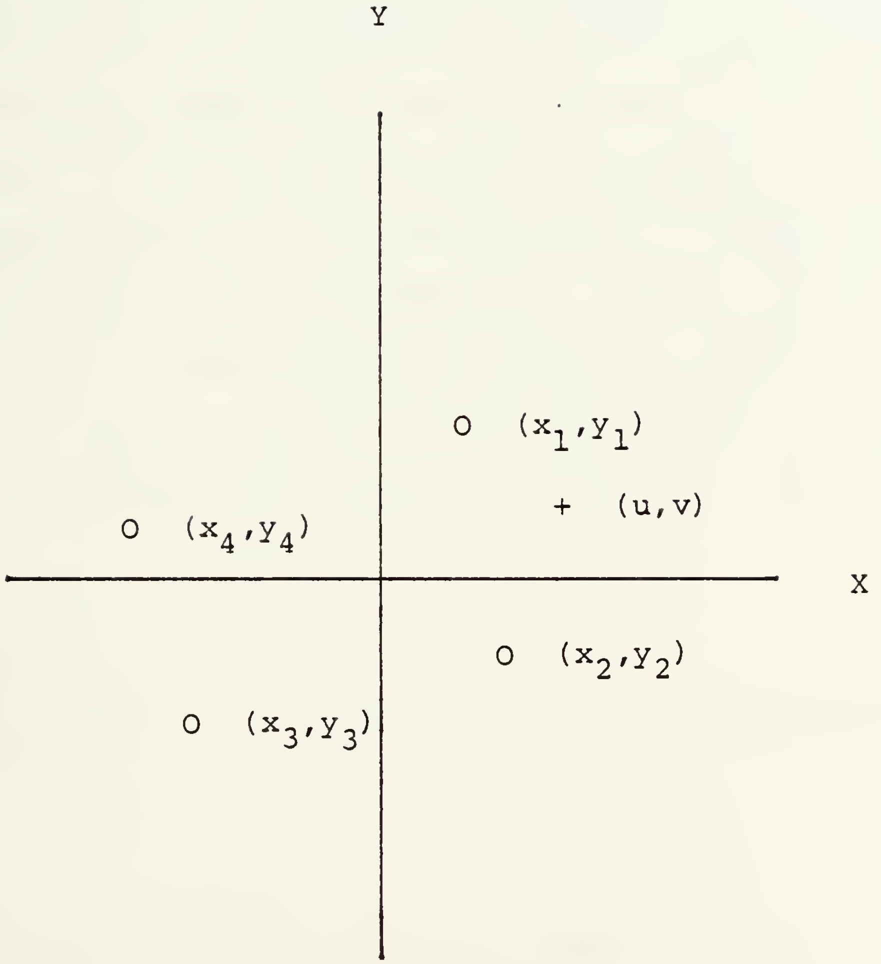


Figure 1

and v, and then subtracting it from unity:

$$p = 1 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1 - p(u,v)\}^n f_1(u,v)\,du\,dv \, . \tag{1.2}$$

B. UPPER BOUND ON THE SALVO KILL PROBABILITY

In the example mentioned earlier, Morse and Kimball observed that the probability of destroying the track by dropping two bombs in salvo is always smaller than the destruction probability of Case (3), namely the case in which the two bombs are dropped independently on separate passes. Similar observations are also pointed out in the review paper by Eckler and Burr. The observation is valuable, but there has been no proof given to this fundamental property discovered numerically. Indeed, we have the following proposition without any specific assumptions on $f_1(u,v)$, $f_2(x,y)$ and $D(x,y)$.

Proposition 1.1

Let the salvo kill probability with n weapons be denoted as P_n

$$P_n = 1 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1 - p(u,v)\}^n f_1(u,v)\,du\,dv \, . \tag{1.3}$$

Then,

$$P_n < 1 - (1 - P_1)^n . \tag{1.4}$$

Namely, salvo P_n is always smaller than the kill probability of n independent tries.

Proof

The proof is immediate from a theorem in the book by Hardy and others [Ref. 5]. Define a mean M_r of a non-negative function g(x) for a positive r as,

$$M_r(g) = \{ \int g^r f \, dx \}^{1/r} ,$$

where f = f(x) is a weighting function, positive everywhere and,

$$\int f \, dx = 1 .$$

Theorem 192 from Ref. 5 tells us that if $0 < r < s$ and $M_s(g)$ is finite, then

$$M_r < M_s ,$$

unless g(x) is a constant.

The weighting function f corresponds to $f_1(u,v)$ in our proposition, and g corresponds to our $1 - p(u,v)$. Therefore, for $n > 1$, we have

$$M_1(1-p) < M_n(1-p)$$

or

$$\{M_1(1-p)\}^n < \{M_n(1-p)\}^n ,$$

which means

$$(1-P_1)^n < 1 - P_n . \qquad \square$$

Here is an outline of the succeeding chapters: In Chapter II, we investigate the simplest salvo model which is characterized by circular normal errors and circular Gaussian damage function. After that, an approximate formula for the optimal ballistic dispersion is introduced in Chapter III; we regard the material in Chapter III as the "core" of this dissertation. In Chapter IV, the approximate formula is applied to the salvo model of Chapter II, and its accuracy is studied. In Chapter V, an approximate method of calculating the salvo kill probability of the salvo model with the general damage function is presented; this case is more difficult than the one where the damage function is circular normal, so we resort to further approximations. In Chapter VI, we investigate the salvo model with the so-called cookie cutter damage function, relying heavily on the results of Chapters III and V. Finally, salvo model without circular symmetry is discussed in Chapter VII. Our concern throughout

is the optimization of artificial dispersion. In some cases no closed form solution is available, so we offer efficient methods for the computation or approximation of salvo kill probability with a given dispersion. In other cases we are able to offer closed form solutions.

II. SIMPLE SALVO MODEL

In this chapter we deal with the simplest salvo model with circular normal errors and a circularly symmetric Gaussian damage function.

Let us adopt a Cartesian coordinate system (x,y) the origin of which coincides with the center of impact of the n weapons. The target position with respect to this coordinate system is denoted as U and V in the x and y direction, respectively. (U,V) is a random bias common to all the n weapons. In this chapter (U,V) is assumed as a circular normal variate; namely, we assume that U and V are independent and identically distributed normal random variables with mean 0 and variance σ_1^2. The joint density of U and V is, therefore

$$f_1(u,v) = \frac{1}{2\pi\sigma_1^2} e^{-\frac{u^2+v^2}{2\sigma_1^2}} . \qquad (2.1)$$

The impact point of the ith weapon is denoted as (X_i,Y_i). It is assumed that (X_i,Y_i), $i = 1,2,\ldots,n$ are independent and identically distributed circular normal random variables with joint density function

$$f_2(x,y) = \frac{1}{2\pi\sigma_2^2} e^{-\frac{x^2+y^2}{2\sigma_2^2}} . \qquad (2.2)$$

As to the damage function, we assume that the so-called Gaussian damage function with circular symmetry: If the target is at (u,v) and a weapon impacts at (x,y), then the destruction probability is given by

$$D(u-x,v-y) \;=\; e^{-\frac{(u-x)^2+(v-y)^2}{2\alpha^2}} . \tag{2.3}$$

In the following, let us call this model the Simple Salvo Model.

According to Grubbs [Ref. 6], the Simple Salvo Model was first studied as early as in 1953 by H. K. Weiss in BRL Report No. 879, "Methods for Computing the Effectiveness of Area Weapons." The formula for the salvo kill probability (2.7) given in the next section is attributed to him. Later, Breaux [Ref. 7] found that this formula was not suitable for computation when n is large, and gave another method of computation. However, it seems that there is no published research investigating this model in full depth; the studies usually come to an end when expressions for the kill probability are derived. In this chapter, we investigate first the property of the kill probability as a function of parameters involved in the model. Methods for computing the kill probability and bounds for the kill probability are also dealt with in the later sections.

A. SALVO KILL PROBABILITY AS A FUNCTION OF n, σ_1, σ_2, AND α

The salvo kill probability is derived in two steps as is stated in Chapter I. First the conditional kill probability by the ith weapon is calculated given that the random bias (U,V) takes a value (u,v).

$$p(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(u-x,v-y) f_2(x,y)\,dxdy \ . \tag{1.1}$$

The salvo kill probability is then obtained by averaging $1-\{1-p(u,v)\}^n$ over the distribution of (U,V), as

$$p = 1 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1-p(u,v)\}^n f_1(u,v)\,dudv \ . \tag{1.2}$$

Using the assumptions (2.2) and (2.3), we easily calculate the conditional kill probability (1.1)

$$p(u,v) = \frac{1}{2\pi\sigma_2^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{(u-x)^2+(v-y)^2}{2\alpha^2} - \frac{x^2+y^2}{2\sigma_2^2}} dxdy$$

and get

$$p(u,v) = \frac{\alpha^2}{\alpha^2 + \sigma_2^2}\, e^{-\frac{u^2+v^2}{2(\alpha^2+\sigma_2^2)}} \ . \tag{2.4}$$

Substituting (2.1) and (2.4) into (1.2), the salvo kill probability is

$$P = 1 - \frac{1}{2\pi\sigma_1^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{1 - \frac{\alpha^2}{\alpha^2+\sigma_2^2} e^{-\frac{u^2+v^2}{2(\alpha^2+\sigma_2^2)}}\}^n e^{-\frac{u^2+v^2}{2\sigma_1^2}} dudv .$$

Conversion from Cartesian coordinates into polar coordinates gives

$$P = 1 - \frac{1}{\sigma_1^2}\int_0^{\infty}\{1 - \frac{\alpha^2}{\alpha^2+\sigma_2^2} e^{-\frac{r^2}{2(\alpha^2+\sigma_2^2)}}\}^n e^{-\frac{r^2}{2\sigma_1^2}} r\,dr .$$

From this we get

$$P = 1 - \int_0^{\infty}(1 - \frac{\lambda}{\rho} e^{-t/\rho})^n e^{-t} dt \qquad (2.5)$$

where

$$\lambda = \alpha^2/\sigma_1^2, \qquad \rho = (\alpha^2 + \sigma_2^2)/\sigma_1^2 . \qquad (2.6)$$

There are 4 parameters in our model, but the salvo kill probability is determined by three, n, σ_1/α, and σ_2/α, or n, λ, and ρ as is seen in (2.5).

If the integrand in (2.5) is expanded in a binomial series and integrated term by term, we obtain the Weiss

formula for the salvo kill probability.

$$P = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \left(\frac{\lambda}{\rho}\right)^{j} \frac{\rho}{j+\rho} . \qquad (2.7)$$

P is obviously an increasing function of n, and is decreasing in σ_1^2/α^2. The latter is easily verified by inspecting (2.5), where λ/ρ is constant for fixed σ_2^2/α^2. The σ_2^2/α^2-dependence is, however, not obvious. In (2.5), λ is constant for a fixed value of σ_1^2/α^2, and there are two factors which involve ρ in the integrand, with ρ tending to increase P in one and to decrease P in the other.

Table 1 and Table 2 are given here to show the σ_1^2/α^2- and σ_2^2/α^2-dependence of the salvo kill probability, where n is kept constant, n = 2 for Table 1 and n = 16 for Table 2. For fixed n and σ_2^2/α^2, P is decreasing in σ_1^2/α^2 as mentioned above. However, the σ_2^2/α^2-dependence is a little different. It is observed that, generally, P is decreasing in σ_2^2/α^2 for small values of σ_1^2/α^2, but for a larger σ_1^2/α^2, P increases when σ_2^2/α^2 is increased, reaches a maximum value, and then decreases. It is also noted that from the viewpoint of real world applications, the case of large σ_1^2/α^2 with large n is especially interesting; e.g., as seen in Table 2, the kill probability for n = 16 and $\sigma_1^2/\alpha^2 = 8.0$ is improved to 0.5881 when σ_2^2/α^2 is increased to a suitable value. The maximum P is more than 1.7 times the original value for $\sigma_2^2/\alpha^2 = 0$.

Table 1. Salvo Kill Probability

$n = 2$

	σ_1^2/α^2					
σ_2^2/α^2	0.00	0.50	1.00	2.00	4.00	8.00
0.0	1.0000	0.8333	0.6667	0.4667	0.2889	0.1634
0.1	0.9917	0.8171	0.6591	0.4669	0.2923	0.1666
0.2	0.9722	0.7977	0.6487	0.4647	0.2940	0.1689
0.3	0.9467	0.7767	0.6365	0.4609	0.2946	0.1706
0.4	0.9184	0.7550	0.6232	0.4560	0.2944	0.1717
0.5	0.8889	0.7333	0.6095	0.4502	0.2935	0.1724
0.6	0.8594	0.7120	0.5956	0.4439	0.2920	0.1728
0.7	0.8304	0.6912	0.5818	0.4373	0.2902	0.1730
0.8	0.8025	0.6712	0.5681	0.4305	0.2881	0.1729
0.9	0.7756	0.6518	0.5547	0.4236	0.2858	0.1726
1.0	0.7500	0.6333	0.5417	0.4167	0.2833	0.1722

Table 2 Salvo Kill Probability

$n = 16$

	σ_1^2/α^2					
σ_2^2/α^2	0.00	0.50	1.00	2.00	4.00	8.00
0.00	1.0000	0.9935	0.9412	0.7835	0.5511	0.3370
0.25	1.0000	0.9956	0.9570	0.8234	0.6013	0.3784
0.50	1.0000	0.9965	0.9659	0.8504	0.6399	0.4128
0.75	1.0000	0.9968	0.9710	0.8692	0.6700	0.4417
1.00	1.0000	0.9967	0.9739	0.8824	0.6938	0.4662
1.25	0.9999	0.9962	0.9752	0.8915	0.7126	0.4871
1.50	0.9997	0.9954	0.9753	0.8978	0.7276	0.5049
1.75	0.9993	0.9941	0.9745	0.9017	0.7394	0.5201
2.00	0.9985	0.9922	0.9728	0.9038	0.7485	0.5330
2.25	0.9972	0.9899	0.9704	0.9045	0.7555	0.5440
2.50	0.9954	0.9869	0.9672	0.9038	0.7606	0.5534
2.75	0.9930	0.9833	0.9635	0.9021	0.7642	0.5612
3.00	0.9900	0.9792	0.9591	0.8996	0.7665	0.5678
3.25	0.9863	0.9745	0.9541	0.8962	0.7676	0.5732
3.50	0.9821	0.9692	0.9486	0.8921	0.7677	0.5776
3.75	0.9772	0.9635	0.9427	0.8875	0.7670	0.5810
4.00	0.9719	0.9573	0.9364	0.8824	0.7655	0.5837
4.25	0.9660	0.9507	0.9298	0.8769	0.7634	0.5857
4.50	0.9597	0.9438	0.9228	0.8710	0.7607	0.5870
4.75	0.9530	0.9366	0.9155	0.8648	0.7576	0.5878
5.00	0.9459	0.9291	0.9081	0.8583	0.7541	0.5881
5.25	0.9386	0.9214	0.9004	0.8516	0.7502	0.5880
5.50	0.9309	0.9135	0.8926	0.8448	0.7460	0.5874
5.75	0.9231	0.9055	0.8847	0.8378	0.7416	0.5865
6.00	0.9151	0.8973	0.8767	0.8307	0.7369	0.5853

Experience shows that there is one and only one local maximum of P as a function of σ_2^2/α^2 for fixed n and σ_1^2/α^2, but we have not succeeded in obtaining a proof.

The salvo kill probability can be increased by increasing σ_2^2/α^2 from zero for given n and σ_1^2/α^2 if the condition below is satisfied.

Proposition 2.1

The salvo kill probability P for fixed n, σ_1 and α is not maximal at $\sigma_2 = 0$ if and only if

$$\frac{1}{1+\lambda} + \frac{1}{2+\lambda} + \dots + \frac{1}{n+\lambda} > 1 , \qquad (2.8)$$

where

$$\lambda = \alpha^2/\sigma_1^2 .$$

Proof

We will show that

$$dP/d\rho \Big|_{\rho=\lambda} > 0 \qquad (2.9)$$

is equivalent to (2.8). Using (2.7), (2.9) is written as

$$n \int_0^\infty (1-e^{-t/\lambda})^{n-1} e^{-t/\lambda-t} dt < n \int_0^\infty (1-e^{-t/\lambda})^{n-1} e^{-t/\lambda-t} t/\lambda \, dt \qquad (2.10)$$

Now define

$$I(x) = n \int_0^\infty (1 - e^{-t/\lambda})^{n-1} e^{-t/\lambda - xt}\, dt .$$

Then the inequality (2.10) is rewritten as

$$I(1) < -I'(1)/\lambda , \tag{2.11}$$

where I'(1) means the value of the derivative dI/dx at $x = 1$. Let $s = \exp(-t/\lambda)$ in the integral, then $I(x)$ is

$$I(x) = n\lambda \int_0^1 s^{\lambda x}(1-s)^{n-1}\, ds$$

$$= \lambda \frac{n}{\lambda x+1} \frac{n-1}{\lambda x+2} \cdots \frac{1}{\lambda x+n} .$$

Therefore,

$$\frac{I'(x)}{I(x)} = - \sum_{j=1}^{n} \frac{\lambda}{\lambda x+j} .$$

The inequality (2.8) is readily obtained from this equation with $x = 1$ and (2.11). □

Proposition 2.2

A sufficient condition that the salvo kill probability P for fixed n, σ_1, and α is not maximal at $\sigma_2 = 0$ is

$$\alpha^2/\sigma_1^2 \leqq n/(e-1) - 1 . \qquad (2.12)$$

Proof

Since $1/(x+\lambda)$ is decreasing in x,

$$\sum_{j=1}^{n} \frac{1}{j+\lambda} > \sum_{j=1}^{n} \int_{j}^{j+1} \frac{dx}{x+\lambda} = \int_{1}^{n+1} \frac{1}{x+\lambda}\, dx = \ln \frac{n+1+\lambda}{1+\lambda} .$$

If the right-hand-side is equal to or greater than unity, i.e., $1+n/(1+\lambda) \geqq e$, then the condition (2.8) in Proposition 2.1 is satisfied. The relation (2.12) is immediate from the above-mentioned inequality. □

Proposition 2.3

Assume that (2.8) holds. A necessary condition that the salvo kill probability be maximum at some point $\sigma_2 > 0$ for fixed n, σ_1 and α is

$$\sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} j \left(\frac{\lambda}{\rho}\right)^{j-1} \frac{\rho+j-1}{(\rho+j)^2} = 0 , \qquad (2.13)$$

where

$$\lambda = \alpha^2/\sigma_1^2 , \qquad \rho = (\alpha^2 + \sigma_2^2)/\sigma_1^2 .$$

Proof

Differentiate (2.7) with respect to ρ, and equat it to 0. The equation (2.13) is immediate. □

Example. Case n = 2

Equation (2.13) is generally too complicated to be solved for σ_2/α in closed form. Here we solve it for the simplest case, n = 2.

The inequality (2.10) with n = 2 is

$$\frac{1}{1+\lambda} + \frac{1}{2+\lambda} > 1 ,$$

and after a bit of algebra, this reduces to

$$1/\lambda = \sigma_1^2/\alpha^2 > 2/(\sqrt{5} - 1) . \tag{2.14}$$

Now, assume that σ_1^2/α^2 is large and satisfies (2.14). From (2.13)

$$2 \frac{\rho}{(\rho+1)^2} - 2 \frac{\lambda}{\rho} \frac{\rho+1}{(\rho+2)^2} = 0 .$$

Therefore,

$$\begin{aligned} \sigma_2^2/\alpha^2 &= \rho/\lambda - 1 = \frac{(\rho+1)^3}{\rho(\rho+2)^2} - 1 , \\ \sigma_1^2/\alpha^2 &= 1/\lambda = \frac{(\rho+1)^3}{\rho^2(\rho+2)^2} . \end{aligned} \tag{2.15}$$

This is a parametric expression of maximizing σ_2^2/α^2 as a function of σ_1^2/α^2 the curve of which is shown in Fig. 2. The value of the parameter ρ which corresponds to the point

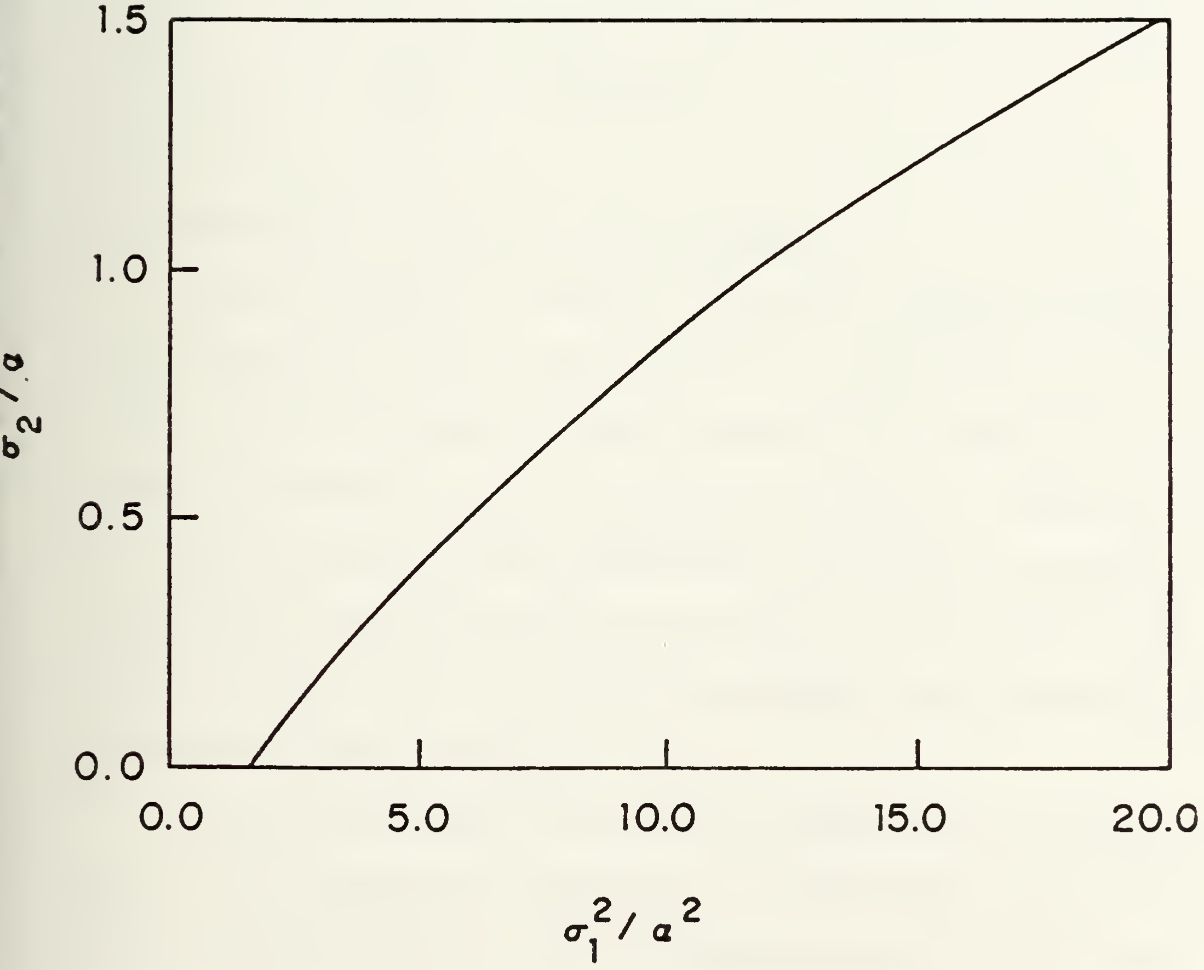


Figure 2.

$(\sigma_1^2/\alpha^2 = (\sqrt{5}+1)/2,\ \sigma_2^2/\alpha^2 = 0)$ is $\rho = \rho_0$, where

$$\rho_0 = (\sqrt{5} - 1)/2 .$$

When ρ approaches zero,

$$\sigma_1^2/\alpha^2 \sim 1/4\rho^2 , \qquad \sigma_2^2/\alpha^2 \sim 1/4\rho ,$$

and therefore, the asymptotic form of the curve is

$$\sigma_2^2/\alpha^2 = \frac{1}{2}\,\sigma_1/\alpha . \tag{2.16}$$

B. COMPUTING THE SALVO KILL PROBABILITY

In the preceding section, the Weiss formula for the salvo kill probability was given in (2.7). According to Breaux [Ref. 7], this formula is not suitable for calculation for a large n because it is an alternating series. Breaux pointed out that the salvo kill probability could be expressed in terms of an Incomplete Beta Function, and that for large n, Tang's method [Ref. 8] would be advisable for calculating an Incomplete Beta Function.

First, we present a recurrence formula for calculating the salvo kill probability which is, in principle, the same as that of Tang's method. Its generalization will be given in a later chapter.

Proposition 2.4

Let us denote the salvo miss probability by n weapons as $Q(n) = 1-P$.

$$Q(n) = \int_0^\infty (1 - \mu e^{-t/\rho})^n \, e^{-t} \, dt \, . \tag{2.17}$$

Then $Q(n)$ satisfies the following recurrence equations:

$$Q(n) = \frac{n}{n+\rho} Q(n-1) + \frac{\rho}{n+\rho}(1-\mu)^n \, , \quad n = 1,2,\ldots \tag{2.18}$$

$$Q(0) = 1 \, , \tag{2.19}$$

where

$$\mu = \lambda/\rho = \alpha^2/(\alpha^2 + \sigma_2^2) \, , \quad \rho = (\alpha^2 + \sigma_2^2)/\sigma_1^2 \, .$$

Proof

From (2.17)

$$Q(n) = \int_0^\infty (1 - \mu e^{-t/\rho})^n \, e^{-t} \, dt$$

$$= \int_0^\infty (1-\mu e^{-t/\rho})^{n-1} e^{-t} dt - \mu \int_0^\infty (1-\mu e^{-t/\rho})^{n-1} e^{-t/\rho-t} dt$$

The first term in the right hand side is $Q(n-1)$. In the second term, we integrate it by parts, and have

$$Q(n) = Q(n-1) - \frac{\rho}{n}(1-\mu e^{-t/\rho})^n e^{-t}\Big|_0^\infty - \frac{\rho}{n}\int_0^\infty (1-\mu e^{-t/\rho})^n e^{-t}\, dt$$

$$= Q(n-1) + \frac{\rho}{n}(1-\mu)^n - \frac{\rho}{n}\, Q(n) \;.$$

The equation (2.18) is immediate. □

It is worthwhile to note that the method presented in Proposition 2.4 gives not only the value of $Q(n)$ for a specified n_0, but also gives all the $Q(n)$'s up to n_0. It is an important advantage of this method over the Weiss formula because we are often interested in the salvo kill probabilities for several n's. As is shown in Table 3, the computational time for $Q(n)$ by our recurrence algorithm is about twice the time required for computation of a single $Q(n)$ using the Weiss formula. When the salvo kill probabilities for more than a single n are needed, therefore, use of the recurrence algorithm given in Proposition 2.4 is recommended.

We have another efficient algorithm for computing the salvo kill probability, which is good even for large n.

Table 3. Comparison of the Processing Time for Computing the Salvo Kill Probability

Mean time (microsecond), $\rho = 1$, and $\mu = 0.5$.

n	Weiss formula	Recurrence formula
10	92	175
20	189	383
30	300	587

Mean time (microsecond), $\rho = 1$, and $n = 10$.

μ	Weiss formula	Recurrence formula
0.1	90	196
0.5	92	175
0.9	100	186

Computation is on the IBM 3033 installed in the Naval Postgraduate School. Each value is the mean of 4 trials. One trial is 1000 repetitions of calculation, and the elapsed time units are divided by 1000 and multiplied by 26 (microseconds).

Proposition 2.5

The salvo miss probability Q(n) by n weapons

$$Q(n) = \int_0^\infty (1 - \mu e^{-t/\rho})^n e^{-t} dt \qquad (2.17)$$

satisfies the following relationship

$$Q(n) = \sum_{k=0}^{n} \binom{n}{k} (1-\mu)^{n-k} \mu^k q(k) , \qquad (2.20)$$

where q(k) is the salvo miss probability by k weapons with $\mu = 1$,

$$q(k) = \int_0^\infty (1 - e^{-t/\rho})^k e^{-t} dt .$$

Proof

$$Q(n) = \int_0^\infty \{(1 - \mu) + \mu(1 - e^{-t/\rho})\}^n e^{-t} dt$$

$$= \sum_{k=0}^{n} \binom{n}{k} (1-\mu)^{n-k} \mu^k \int_0^\infty (1 - \mu e^{-t/\rho})^k e^{-t} dt .$$

□

Corollary 2.6

The salvo miss probability (2.17) can be calculated by the formula

$$Q(n) = \sum_{k=0}^{n} A_k , \tag{2.21}$$

where

$$A_k/A_{k-1} = \mu(n-k+1)/(1-\mu)(k+\rho) , \quad k = 1,2,\ldots$$

$$A_0 = (1-\mu)^n . \tag{2.22}$$

Proof

As can be seen from (2.18) and (2.19) with $\mu = 1$, $q(0)$ is unity, and $q(k)/q(k-1)$ is $k/(k+\rho)$. Utilizing the fact that $\binom{n}{k}/\binom{n}{k-1} = (n-k+1)/k$, the corollary can be obtained by substitution. □

Note that all the terms in the expression (2.24) are positive, so there is no problem arising from cancellation of terms with alternating signs as was observed in the Weiss formula (2.7).

In checking the computation, the following P's for special cases would be of use.

(1) The case $\sigma_1 = 0$ corresponds to $\rho \to \infty$, and we have

$$P = 1 - (1-\mu)^n .$$

(2) The case $\sigma_2 = 0$ corresponds to $\mu = 1$, $\rho = \lambda$, and so from the recurrence formula, we have

$$P = 1 - \frac{n(n-1)\cdots 1}{(n+\lambda)(n-1+\lambda)\cdots(1+\lambda)} . \tag{2.24}$$

(3) The case $\alpha^2 + \sigma_2^2 = \sigma_1^2$. The formula (2.5) with $\rho = 1$ is easily integrated to give

$$P = 1 - \frac{\lambda}{n+1}\{1 - (1-\lambda)^{n+1}\} , \tag{2.25}$$

where

$$\mu = \alpha^2/(\alpha^2 + \sigma_2^2) , \quad \lambda = \alpha^2/\sigma_1^2 .$$

C. BOUNDS TO THE SALVO KILL PROBABILITY

In Chapter I it was pointed out that an upper bound of the salvo kill probability is given by the kill probability of n repeated independent shots.

$$P < \overline{P}$$

where

$$\overline{P} = 1 - [\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{1 - p(u,v)\}f_1(u,v)\,du\,dv]^n .$$

Under the assumptions of the present chapter, we have

Proposition 2.7

An upper bound to the salvo kill probability is given by

$$\overline{P} = 1 - \left(1 - \frac{\lambda}{1+\rho}\right)^n , \qquad (2.26)$$

where

$$\lambda = \alpha^2/\sigma_1^2 , \quad \rho = (\alpha^2 + \sigma_2^2)/\sigma_1^2 .$$

As to a lower bound, we have

Proposition 2.8 (Merritt and King)

A lower bound to the salvo kill probability is given by

$$\underline{P} = 1 - \rho(n\mu)^{-\rho} \, \Gamma_{n\mu}(\rho) , \qquad (2.27)$$

where $\Gamma_x(\rho)$ is the Incomplete Gamma Function,

$$\Gamma_x(\rho) = \int_0^x e^{-t} t^{\rho-1} \, dt ,$$

and

$$\mu = \alpha^2/(\alpha^2 + \sigma_2^2) , \quad \rho = (\alpha^2 + \sigma_2^2)/\sigma_1^2 .$$

Table 4 and Table 5 show how the true kill probabilities are bracketed by these two bounds. The triplets of entries

Table 4 Salvo Kill Probability and Its Bounds

$n = 2$

σ_2^2/α^2	σ_1^2/α^2 0.00	0.50	1.00	2.00	4.00	8.00
	0.8889	0.7500	0.6400	0.4898	0.3306	0.1994
0.5	0.8889	0.7333	0.6095	0.4502	0.2935	0.1724
	0.7364	0.6187	0.5216	0.3911	0.2580	0.1527
	0.7500	0.6400	0.5556	0.4375	0.3056	0.1900
1.0	0.7500	0.6333	0.5417	0.4167	0.2833	0.1722
	0.6321	0.5443	0.4715	0.3679	0.2532	0.1552
	0.6400	0.5556	0.4898	0.3951	0.2840	0.1814
1.5	0.6400	0.5524	0.4825	0.3829	0.2696	0.1689
	0.5507	0.4832	0.4267	0.3429	0.2442	0.1542
	0.5556	0.4898	0.4375	0.3600	0.2653	0.1736
2.0	0.5556	0.4881	0.4333	0.3524	0.2554	0.1643
	0.4866	0.4333	0.3882	0.3192	0.2338	0.1516
	0.4898	0.4375	0.3951	0.3306	0.2489	0.1664
2.5	0.4898	0.4365	0.3925	0.3255	0.2418	0.1593
	0.4353	0.3922	0.3554	0.2977	0.2233	0.1482
	0.4375	0.3951	0.3600	0.3056	0.2344	0.1597
3.0	0.4375	0.3944	0.3583	0.3021	0.2292	0.1542
	0.3935	0.3580	0.3274	0.2784	0.2131	0.1444
	0.3951	0.3600	0.3306	0.2840	0.2215	0.1536
3.5	0.3951	0.3596	0.3294	0.2815	0.2175	0.1492
	0.3588	0.3291	0.3032	0.2611	0.2034	0.1404
	0.3600	0.3306	0.3056	0.2653	0.2099	0.1479
4.0	0.3600	0.3303	0.3048	0.2635	0.2068	0.1443
	0.3297	0.3045	0.2823	0.2457	0.1944	0.1365

Triplets are, from the top, $\overline{P}$, the upper bound, P, the true value, and $\underline{P}$, the lower bound.

Table 5 Salvo Kill Probability and Its Bounds

$n = 16$

	σ_1^2/α^2					
σ_2^2/α^2	0.00	0.50	1.00	2.00	4.00	8.00
1.0	1.0000	0.9997	0.9985	0.9900	0.9459	0.8147
	1.0000	0.9967	0.9739	0.8824	0.6938	0.4662
	0.9997	0.9944	0.9688	0.8750	0.6867	0.4611
2.0	0.9985	0.9954	0.9900	0.9719	0.9151	0.7824
	0.9885	0.9922	0.9728	0.9038	0.7485	0.5330
	0.9952	0.9862	0.9644	0.8935	0.7388	0.5258
3.0	0.9900	0.9821	0.9719	0.9459	0.8819	0.7515
	0.9900	0.9792	0.9591	0.8996	0.7665	0.5678
	0.9817	0.9685	0.9469	0.8864	0.7546	0.5590
4.0	0.9719	0.9597	0.9459	0.9151	0.8481	0.7222
	0.9719	0.9573	0.9364	0.8824	0.7655	0.5837
	0.9592	0.9432	0.9215	0.8674	0.7522	0.5739
5.0	0.9459	0.9309	0.9151	0.8819	0.8147	0.6945
	0.9459	0.9291	0.9081	0.8583	0.7541	0.5881
	0.9305	0.9130	0.8917	0.8424	0.7402	0.5778
6.0	0.9151	0.8987	0.8819	0.8481	0.7824	0.6684
	0.9151	0.8973	0.8767	0.8307	0.7369	0.5853
	0.8983	0.8802	0.8597	0.8145	0.7229	0.5749
7.0	0.8819	0.8650	0.8481	0.8147	0.7515	0.6439
	0.8819	0.8639	0.8441	0.8016	0.7167	0.5780
	0.8647	0.8468	0.8273	0.7857	0.7030	0.5677
8.0	0.8481	0.8313	0.8147	0.7824	0.7222	0.6209
	0.8481	0.8305	0.8117	0.7724	0.6952	0.5680
	0.8310	0.8136	0.7953	0.7570	0.6819	0.5579

Triplets are, from the top, $\overline{P}$, the upper bound, P, the true value, and $\underline{P}$, the lower bound.

in the table are, from the above, the upper bound $\overline{P}$, true value P, and $\underline{P}$, the lower bound. It is observed that $\overline{P} - P$ is not small for smaller values of σ_2^2/α^2, but gets smaller when σ_2^2/α^2 is increased. If σ_1^2/α^2 is small, $\overline{P}$ and P are close. On the other hand, $\underline{P}$ is fairly close to P over wide ranges of σ_1^2/α^2 and σ_2^2/α^2. Figures 3 and 4 illustrate this. It seems reasonable that Merritt and King used $\underline{P}$ as an approximate formula for P in their study on the optimal ballistic dispersion, because computation was really a problem at that time. Nowadays, however, direct computation of P is much more natural.

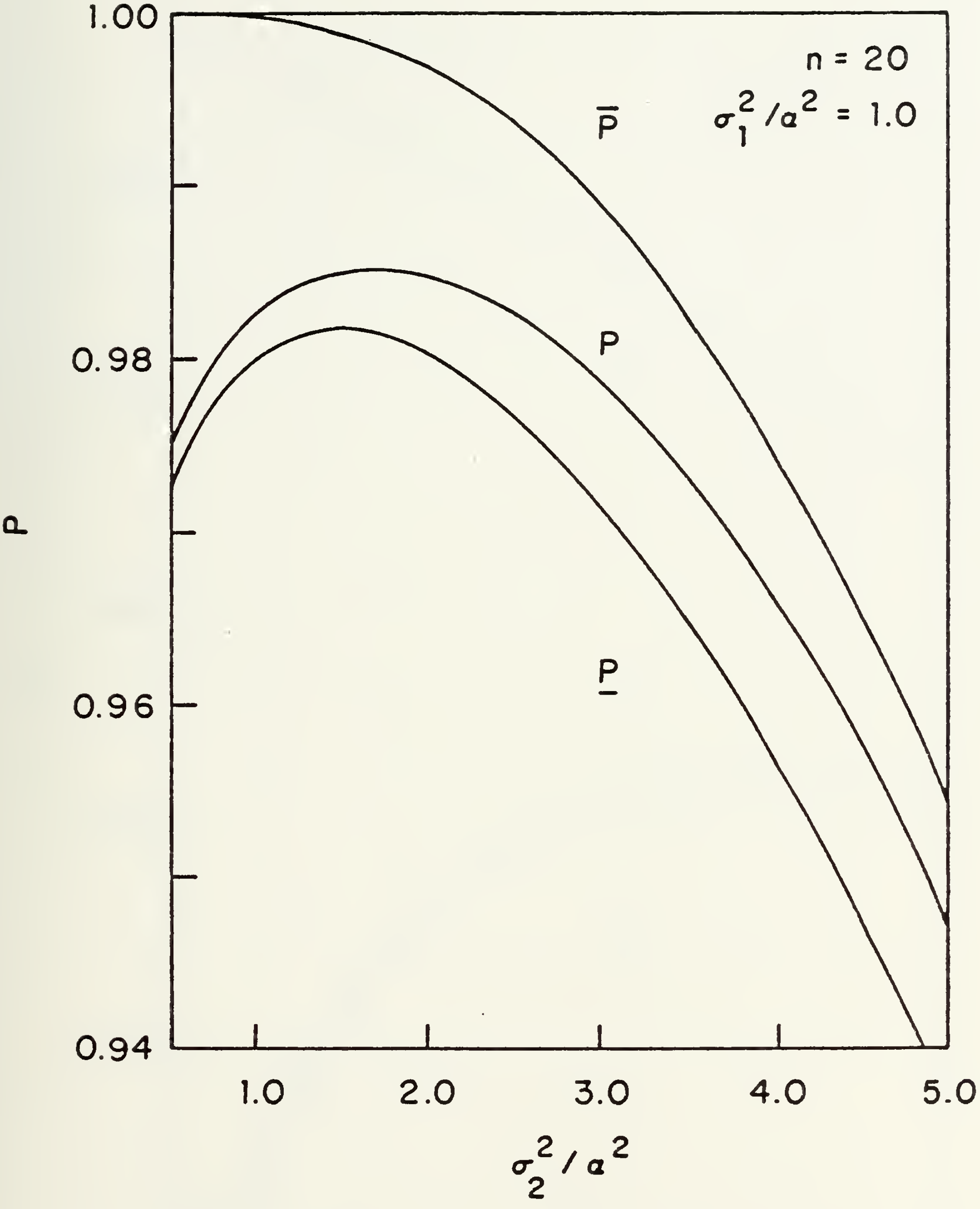
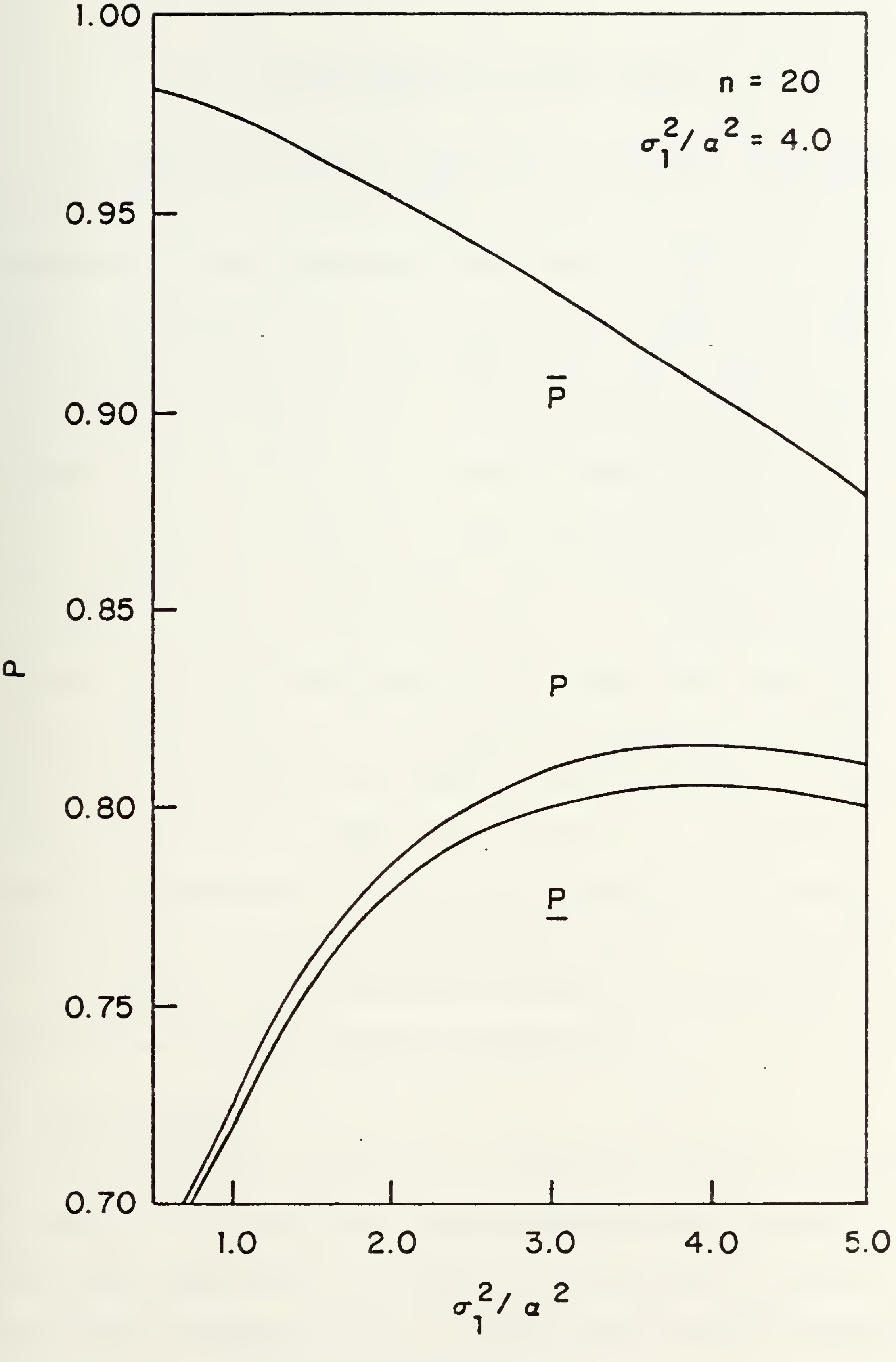

1.00
0.98
0.96
0.94
P
1.0
2.0
3.0
4.0
5.0
σ_2^2/a^2
n = 20
$\sigma_1^2/a^2 = 1.0$
$\overline{P}$
P
$\underline{P}$

Figure 3.



1.00
0.95
0.90
0.85
0.80
0.75
0.70
P
n = 20
$\sigma_1^2/a^2 = 4.0$
$\bar{P}$
P
$\underline{P}$
1.0
2.0
3.0
4.0
5.0
σ_1^2/a^2

Figure 4.

III. OPTIMIZATION OF SALVO FIRING

In Chapter II we have learned the following: The salvo kill probability is a decreasing function of σ_1 if other parameters are kept constant. Therefore, to obtain higher salvo kill probability, it is necessary to make σ_1 as small as possible. The salvo kill probability as a function of σ_2 is, however, complicated. If σ_1 is sufficiently small, the salvo kill probability is monotone decreasing in σ_2, but if σ_1 is not so small and satisfies the condition (2.8), then the salvo kill probability is not maximal at $\sigma_2 = 0$, but takes a maximum value at some $\sigma_2 > 0$. In other words, the smallest ballistic dispersion is not always most desirable. Deliberate increase of the ballistic dispersion is encountered in shotguns and in related military weapons. In this chapter we present an approximate formula for the optimal ballistic dispersion. We will first briefly sketch Walsh's theory [Ref. 9] since our approximate formula is based on it. In Section B our approximate formula is presented. Its accuracy will be studied in Chapter IV.

A. WALSH THEORY

The Walsh model is based on assumptions similar to those of Chapter I. A salvo of n weapons is fired at a target. The impact point error is composed of two parts, a random bias which is common to all n weapons, and a round-to-round

error. A Cartesian coordinate system is chosen so that its origin coincides with the center of impact of n weapons. The coordinates of the random position of the target are denoted as U and V with respect to this system. The joint probability density of U and V is given by $f_1(u,v)$. The impact points of weapons (X_i,Y_i) are independent and identically distributed random variables with joint density $f_2(x,y)$. It is assumed that impact of a weapon at (x,y), given the target is at (u,v), destroys the target with probability D(u-x,v-y) independently of other weapons. The lethal area of a weapon is denoted as A:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(x,y)\,dxdy = A .$$

The salvo kill probability is given by (1.2),

$$P = 1 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1 - p(u,v)\}^n f_1(u,v)\,dudv , \qquad (1.2)$$

where p(u,v) is the conditional destruction probability by a single weapon given that the target is at (u,v).

$$p(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(u-x,v-y) f_2(x,y)\,dxdy . \qquad (1.1)$$

Note that p(u,v) can be integrated to give A.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(u,v)\,du dv = A \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_2(x,y)\,dx dy = A . \quad (3.1)$$

The problem is to derive the optimal ballistic dispersion, σ_2^2. Instead of solving the original problem, Walsh solved a revised problem that was different from the original one in that the variable is not σ_2^2 itself but rather the function $p(u,v)$.

Proposition 3.1 (Walsh)

$$\max_{p(\cdot\cdot)} P = 1 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1 - p(u,v)\}^n f_1(u,v)\,du dv \quad (3.2)$$

subject to

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(u,v)\,du\, dv &= A , \\ p(u,v) &\geqq 0 \end{aligned} \quad (3.3)$$

has a solution

$$p^*(u,v) = \begin{cases} 1 - \{\frac{C}{f_1(u,v)}\}^{\frac{1}{n-1}} & (u,v) \in E \\ 0 & (u,v) \in \overline{E} \end{cases} \quad (3.4)$$

where

$$E = \{(u,v) \mid f_1(u,v) \geqq C\} . \tag{3.5}$$

C is positive and is given by

$$\iint_E p^*(u,v)\, du\, dv = A . \tag{3.6}$$

The corresponding salvo kill probability is

$$P^* = CA + \iint_E \{f_1(u,v) - C\}\, du\, dv . \tag{3.7}$$

In case the random target bias is given by a normal distribution, we also have

Corollary 3.2

$$\max_{p(\cdot\cdot)} P = 1 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1 - p(u,v)\}^n f_1(u,v)\, du\, dv \tag{3.8}$$

subject to

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(u,v)\, du\, dv = A , \\ &p(u,v) \geqq 0 , \end{aligned} \tag{3.9}$$

where

$$f_1(u,v) = \frac{1}{2\pi\, \sigma_u \sigma_v} e^{-\frac{u^2}{2\sigma_u^2} - \frac{v^2}{2\sigma_v^2}} \tag{3.10}$$

has the solution

$$p^*(u,v) = \begin{cases} 1 - \exp\{-\phi + \frac{1}{2(n-1)}(\frac{u^2}{\sigma_u^2} + \frac{v^2}{\sigma_v^2})\} & (u,v) \in E \\ 0 & (u,v) \in \bar{E} \end{cases} \quad (3.11)$$

where

$$E = \{(u,v) \mid u^2/\sigma_u^2 + v^2/\sigma_v^2 \leqq 2(n-1)\phi\} .$$

ϕ is positive and is determined by

$$e^{-\phi} - 1 + \phi - A/2\pi(n-1)\sigma_u\sigma_v = 0 . \quad (3.12)$$

The maximum value of the salvo kill probability is given by

$$P^* = 1 - e^{-(n-1)\phi}\{1 + (n-1)(1 - e^{-\phi})\} . \quad (3.13)$$

Proof

The equation (3.11) is readily obtained from (3.4) in Proposition 3.1 with $f_1(u,v)$ given by (3.10) and C,

$$C = \frac{1}{2\pi\,\sigma_u\sigma_v} e^{-(n-1)\phi} . \quad (3.14)$$

Substituting (3.11) in (3.6) and changing variables into (r,θ) by

$$u = \sqrt{2(n-1)}\,\sigma_u r\cos\theta, \quad v = \sqrt{2(n-1)}\,\sigma_v r\sin\theta \qquad (3.15)$$

we have

$$A = \iint_E p^*(u,v)\,du\,dv$$

$$= 2\pi(n-1)\sigma_u\sigma_v \int_0^{\sqrt{\phi}} (1 - e^{-\phi+r^2})\,2r\,dr$$

$$= 2\pi(n-1)\sigma_u\sigma_v(\phi - 1 + e^{-\phi}) .$$

Substituting (3.10) and (3.14) in (3.7) with (3.15),

$$p^* = \frac{A}{2\pi\sigma_u\sigma_v} e^{-(n-1)\phi} + \frac{2\pi}{2\pi\sigma_u\sigma_v}(n-1)\sigma_u\sigma_v \int_0^{\sqrt{\phi}} (e^{-(n-1)r^2} - e^{-(n-1)\phi})\,2r\,dr$$

$$= 1 - e^{-(n-1)\phi}\{1 + (n-1)\phi - A/2\pi\sigma_u\sigma_v\} .$$

This formula together with (3.12) gives (3.13). □

For simplicity, let us call the function p*(u,v) the (Walsh) optimal p function, and the associated salvo kill probability P* the Walsh optimal P. It should be noted that the constraint in the revised problem is weaker than that of the original problem, and so the Walsh optimal P is an upper bound to the maximum salvo kill probability of the original problem. It should also be noted that the Walsh optimal p function is not feasible in the original problem (being non zero on only a finite set) if dispersion is present.

B. APPROXIMATE FORMULA FOR THE OPTIMAL BALLISTIC DISPERSION

Walsh developed the theory stated above, but he did not extend his theory beyond the results (3.4)-(3.7). The important part left by Walsh and subsequent workers is to bridge the gap between the revised problem and the original problem. Walsh got the optimal p function as given in (3.4), whereas we need the optimal density function $f_2(x,y)$. The material in this section bridges that gap.

Let us define three double Fourier transforms as follows:

$$\tilde{p}^*(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\omega_1 u+\omega_2 v)}\, p^*(u,v)\,du\,dv\,,$$

$$\tilde{D}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\omega_1 x+\omega_2 y)}\, D(x,y)\,dx\,dy\,, \tag{3.16}$$

$$\tilde{f}_2(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\omega_1 x+\omega_2 y)}\, f_2(x,y)\,dx\,dy\,.$$

Then from (1.1) we have

$$\tilde{p}^*(\omega_1,\omega_2) = \tilde{D}(\omega_1,\omega_2)\,\tilde{f}_2(\omega_1,\omega_2)\,. \tag{3.17}$$

In the Walsh theory, $D(x,y)$ is a given function, and the Walsh optimal p function is given by Proposition 3.1. Therefore, both $D(\omega_1,\omega_2)$ and $p^*(\omega_1,\omega_2)$ in (3.17) are now known functions. Thus the Fourier transform of the "optimal" $f_2^*(x,y)$ can be calculated by

$$\tilde{f}_2(\omega_1,\omega_2) = \frac{\tilde{p}^*(\omega_1,\omega_2)}{\tilde{D}(\omega_1,\omega_2)} , \qquad (3.18)$$

and we will get $f_2^\star(x,y)$ as its inverse Fourier transform. This might have been the end of Walsh theory, but there are difficulties. First, we do not have any guarantee that $f_2(x,y)$ is a density function, because Walsh solved the restricted optimization problem subject to $p(u,v) \geqq 0$ but not $f_2(x,y) \geqq 0$. Further, even if this problem were solved, there would still remain the problem of feasibility of the f_2 function. Walsh suggests in his paper that it would be possible to obtain a real-world dispersion function $f_2(x,y)$ as the theory requires, at least approximately, by careful design of the ammunition, but this may be asking too much of the manufacturing process. We assume below that the function $f_2(x,y)$ is normal, with only the variance σ_2^2 subject to manufacturing control.

The problem, then, is to determine or approximate the optimal ballistic dispersion σ_2^2, given the damage function $D(x,y)$ and the function $p^*(x,y)$ from (say) Proposition 3.1. In the following, both $D(x,y)$ and $p^*(x,y)$ are assumed to be even functions of x and y respectively, and to have "moments" of all degrees. The lethal area is denoted as A.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xD(x,y)\,dxdy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} yD(x,y)\,dxdy = \cdots = 0 , \qquad (3.19)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xp^*(x,y)\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yp^*(x,y)\,dxdy = \cdots = 0 \, ,$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D(x,y)\,dx\,dy = A \, , \tag{3.20}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^{2k}D(x,y)\,dx\,dy < \infty, \quad k = 1,2, \ldots$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^{2k}p^*(x,y)\,dx\,dy < \infty, \quad k = 1,2, \ldots$$

etc.

For the sake of generality, we actually consider the non-symmetric case where the optimal dispersion may be different in the x and y directions. Let the two dispersions be σ_x^2 and σ_y^2. Then

$$f_2(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\, e^{-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}} \, , \tag{3.21}$$

with the double Fourier transform

$$\tilde{f}_2(\omega_1,\omega_2) = e^{-\frac{\sigma_x^2\omega_1^2}{2} - \frac{\sigma_y^2\omega_2^2}{2}} \, . \tag{3.22}$$

Expanding the exponential function in the double Fourier transform of D(x,y), and integrating term by term, we have

$$\tilde{D}(\omega_1,\omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\omega_1 x+\omega_2 y)} D(x,y)\,dx\,dy$$

$$= \sum_{j=0}^{\infty} \frac{i^j}{j!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\omega_1 x+\omega_2 y)^j D(x,y)\,dx\,dy$$

$$= A[1 - \frac{\omega_1^2}{2} \frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 D(x,y)\,dx\,dy - \frac{\omega_2^2}{2} \frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^2 D(x,y)\,dx\,dy + \ldots]$$

where the third line is obtained because D is even. Therefore, for sufficiently small values of ω_1^2 and ω_2^2,

$$\tilde{D}(\omega_1,\omega_2) \doteq \tilde{D}_0(\omega_1,\omega_2) \qquad (3.23)$$

where

$$\tilde{D}_0(\omega_1,\omega_2) = A \exp [-\frac{\omega_1^2}{2} \frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 D(x,y)\,dx\,dy - \frac{\omega_2^2}{2} \frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^2 D(x,y)\,dx\,dy] . \qquad (3.24)$$

Similarly,

$$\tilde{p}^*(\omega_1,\omega_2) \doteq \tilde{p}_0(\omega_1,\omega_2) \tag{3.25}$$

for small ω_1^2 and ω_2^2, where

$$\tilde{p}_0(\omega_1,\omega_2) = A \exp [- \frac{\omega_1^2}{2} \frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 p^*(x,y)\, dx\, dy$$

$$- \frac{\omega_2^2}{2} \frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^2 p^*(x,y)\, dx\, dy] \, . \tag{3.26}$$

Now we determine two parameters σ_x^2 and σ_y^2 in (3.21) by the relation

$$\tilde{f}_2(\omega_1,\omega_2) = \frac{\tilde{p}_0(\omega_1,\omega_2)}{\tilde{D}_0(\omega_1,\omega_2)} \, . \tag{3.27}$$

Then from (3.22), (3.24), and (3.26) we get an approximate formula for the optimal ballistic dispersion:

$$\sigma_x^2 = [\frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 \{p^*(x,y) - D(x,y)\}\, dx\, dy,\ 0]^+$$

$$\tag{3.28}$$

$$\sigma_y^2 = [\frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^2 \{p^*(x,y) - D(x,y)\}\, dx\, dy,\ 0]^+$$

where

$$[a,0]^+ = \begin{cases} a & \text{if } a \geqq 0 \\ 0 & \text{otherwise} \end{cases} \, . \tag{3.29}$$

Note 1

In this proposition, the approximate formula for the optimal dispersion is derived from the condition that $\tilde{f}_2(\omega_1,\omega_2)$ coincides with $\tilde{f}_2^*(\omega_1,\omega_2)$ for infinitesimally small ω_1^2 and ω_2^2. It would be better if we could derive the approximate formula from the condition that $\tilde{f}_2(\omega_1,\omega_2)$ is nearly equal to $\tilde{f}_2^*(\omega_1,\omega_2)$ over a wider domain of ω_1^2 and ω_2^2 which include (0,0). In the next chapter we will give an idea of improving the approximate formula taking this into account.

Note 2

Generally, approximation (3.23) with (3.24) is accurate only up to the second power of ω_1 and ω_2. If the damage function is of Gaussian type,

$$D(x,y) = e^{-\frac{x^2}{2\alpha_x^2} - \frac{y^2}{2\alpha_y^2}},$$

then

$$\tilde{D}_0(\omega_1,\omega_2) = \tilde{D}(\omega_1,\omega_2)$$

is an exact relationship.

As a corollary, we have the following approximate formula for the model with normal errors and Gaussian damage function.

Consider a salvo firing of n weapons. The density functions of the random bias and the round-to-round ballistic

error are, respectively,

$$f_1(u,v) = \frac{1}{2\pi\,\sigma_u\sigma_v}\, e^{-\frac{u^2}{2\sigma_u^2} - \frac{v^2}{2\sigma_v^2}} , \qquad (3.32)$$

and

$$f_2(x,y) = \frac{1}{2\pi\,\sigma_x\,\sigma_y}\, e^{-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}} , \qquad (3.33)$$

the damage function is given by

$$D(u-x,v-y) = e^{-\frac{(u-x)^2}{2\sigma_u^2} - \frac{(v-y)^2}{2\sigma_v^2}} . \qquad (3.34)$$

Then approximate formulae for the optimal ballistic dispersion for fixed n, σ_u, σ_v, α_x, and α_y are

$$(\sigma_x/\alpha_x)^2 = [\tfrac{1}{2}(n-1)^2\phi^2(\sigma_u/\alpha_x)^3(\phi_v/\alpha_y)-(n-1)(\sigma_u/\alpha_x)^2-1,0]^+$$

$$(3.35)$$

$$(\sigma_y/\alpha_y)^2 = [\tfrac{1}{2}(n-1)^2\phi^2(\sigma_v/\alpha_y)^3(\sigma_u/\alpha_x)-(n-1)(\sigma_v/\alpha_y)^2-1,0]^+$$

where ϕ is determined by the equation

$$e^{-\phi} - 1 + \phi - (\alpha_x/\sigma_u)(\alpha_y/\sigma_v)/(n-1) = 0 . \qquad (3.36)$$

An upper bound to the maximum salvo kill probability is given by

$$\overline{P} = 1 - e^{-(n-1)\phi}\{1 + (n-1)(1 - e^{-\phi})\} . \tag{3.37}$$

The solution to the Walsh problem with the above mentioned assumptions is already given in Corollary 3.2. By use of (3.11), we have

$$\frac{1}{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^2 p^*(u,v)\, du\, dv$$

$$= \frac{\sigma_u^3\sigma_v}{A} 2\pi \int_0^{\sqrt{2(n-1)\phi}} \frac{1}{2} r^2 \{1 - e^{-\phi + \frac{r^2}{2(n-1)}}\}\, r\, dr$$

$$= \frac{\pi\sigma_u^3\sigma_v}{A}\{(n-1)^2\phi^2 - (n-1)A/\pi\sigma_u\sigma_v\}$$

and

$$\frac{1}{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 D(x,y)\, dx\, dy = \alpha_x^2 .$$

Substituting these relations into (3.28), we get the approximate formula for σ_x^2 (3.35). Similar arguments follow for σ_y^2.

IV. SIMPLE SALVO MODEL REVISITED

In this chapter we apply the general approximate formula for the optimal ballistic dispersion obtained in the last chapter to the Simple Salvo Model dealt with in Chapter II and investigate its accuracy.

The assumptions in this chapter are the same as those in Chapter II. We consider a salvo of n weapons against a target. The impact point error is composed of two parts, the random bias and the round-to-round error. The position of the target (U,V) has a circular normal distribution centered at (0,0) with variance σ_1^2. The impact points of the ith weapon (X,Y) are assumed independent and identically distributed circular normal random variables centered at (0,0) with variance σ_2^2. Finally, we assume the circular Gaussian damage function with parameter :

$$f_1(u,v) = \frac{1}{2\pi\sigma_1^2} e^{-\frac{u^2+v^2}{2\sigma_1^2}} , \tag{4.1}$$

$$f_2(x,y) = \frac{1}{2\pi\sigma_2^2} e^{-\frac{x^2+y^2}{2\sigma_2^2}} , \tag{4.2}$$

$$D(u-x,v-y) = e^{-\frac{(u-x)^2+(v-y)^2}{2\alpha^2}} . \tag{4.3}$$

In (3.35), let

$$\begin{cases} \sigma_u = \sigma_v = \sigma_1 \\ \sigma_x = \sigma_y = \sigma_2 \\ \alpha_x = \alpha_y = \alpha \end{cases} \tag{4.4}$$

then we have the following:

An approximate formula for the optimal ballistic dispersion of the Simple Salvo Model is given by

$$\sigma_2^2/\alpha^2 = [\phi^2 z^2 - z - 1, 0]^+ \tag{4.5}$$

where

$$z = (n-1)\sigma_1^2/\alpha^2 = 1/(e^{-\phi} - 1 + \phi) . \tag{4.6}$$

An upper bound to the maximum salvo kill probability is given by

$$\overline{P} = 1 - e^{-(n-1)\phi}\{1 + (n-1)(1 - e^{-\phi})\} . \tag{4.7}$$

The true optimal σ_2^2/α^2 is a function of n and σ_1^2/α^2, but according to (4.5) and (4.6) the approximate σ_2^2/α^2 is determined by $z = (n-1)\sigma_1^2/\alpha^2$, ϕ being a parameter. To obtain an explicit formula for σ_2^2/α^2 in z instead of the parametric form, we have

Lemma 4.1

The series solution to the equation in ϕ,

$$e^{-\phi} - 1 + \phi - u^2/2 = 0 \tag{4.8}$$

is given by

$$\phi^2 = u^2 + u^3/3 + u^4/12 + u^5/60 + \ldots \tag{4.9}$$

Proof

Let the left-hand-side of (4.8) be expanded in ϕ. Then

$$u^2 = \phi^2(1 - \phi/3 + \phi^2/12 - \phi^3/60 + \ldots) .$$

If the terms in the right-hand-side of (4.9) are expanded in ϕ, we have

$$\begin{aligned}
u^2 &= \phi^2 - \phi^3/3 + \phi^4/12 - \phi^5/60 + \ldots \\
u^3/3 &= \phi^3/3 - \phi^4/6 + \phi^5/18 - \ldots \\
u^4/12 &= \phi^4/12 - \phi^5/18 + \ldots \\
u^5/60 &= \phi^5/60 - \ldots
\end{aligned}$$

from which (4.9) is obvious. □

Using the Lemma and the relationship (4.6), we can express σ_2^2/α^2 in (4.5) as a power series in $\sqrt{2/z}$. If we keep only the first three terms in the expansion, we get an alternative approximate formula for the optimal ballistic dispersion which we will denote as σ_a^2.

$$\sigma_a^2/\alpha^2 = \frac{2}{3}\sqrt{z/2} - \frac{5}{6} + \frac{1}{30}\sqrt{2/z} , \qquad (4.10)$$

with

$$z = (n-1)\sigma_1^2/\alpha^2 .$$

In Table 6 two values of the approximate optimal ballistic dispersion are compared. One is calculated with (4.5) and (4.6), the other with (4.10). Agreement seems quite satisfactory, differences appearing at most in the fourth decimal digit. The formula (4.10) needs more terms to be a

Table 6. Comparison of Two Formulae

$z = (n-1)\sigma_1^2/\alpha^2$	σ_2^2/α^2	σ_a^2/α^2
800.0	12.5017	12.5017
400.0	8.5971	8.5971
200.0	5.8367	5.8367
100.0	3.8855	3.8854
50.0	2.5069	2.5067
25.0	1.5335	1.5331
12.5	0.8475	0.8467

σ_2^2/α^2 is calculated by (4.5), (4.6), and
σ_a^2/α^2 is calculated by (4.10).

good approximation for smaller values of z, but later we will see that the case with z less than 25 is not important from the viewpoint of application.

Note

For n = 2, the formula (4.10) gives

$$\sigma_a^2/\alpha^2 = \frac{\sqrt{2}}{3}\sigma_1/\alpha - \frac{5}{6} + \frac{\sqrt{2}}{30}\alpha/\sigma_1 . \qquad (4.11)$$

In Chapter II, the exact formula of the optimal ballistic dispersion was derived for n = 2. The series expansion of σ_2^2/α^2 is

$$\sigma_2^2/\alpha^2 = \frac{1}{2}\sigma_1/\alpha - \frac{3}{4} + \frac{7}{64}\alpha/\sigma_1 - \ldots \qquad (4.12)$$

It is interesting to note that every term in (4.11) is smaller than the corresponding term in (4.12): The approximate value given by (4.11) for n = 2 is always smaller than the true value.

For comparison's sake, we will also consider the classical approximate formula by Merritt and King [Ref. 2] who approximate the optimal ballistic dispersion by $\sigma_c{}^2$ where

$$\sigma_c^2/\alpha^2 = \frac{1}{2}\sqrt{z_c} \qquad (4.13)$$

$$z_c = n\sigma_1^2/\alpha^2 . \qquad (4.14)$$

A. ACCURACY INVESTIGATION

In this section we will investigate the accuracy of both our approximate formula and the classical one.

Table 7A for z = 400, Table 7B for z = 100, and Table 7C for z = 25 illustrate the calculation carried out for the accuracy study. For a fixed z, the approximate value for the optimal ballistic dispersion divided by α^2 is a constant which is given next to the value of z in the table. On the other hand, the true optimal value increases with n: The first column is n and in the second column is given the corresponding true optimal σ_2^2/α^2, the calculation of which is based on the necessary condition of optimality (2.13), and requires fairly long calculation time. The relative error of our approximate formula $\Delta\sigma_a^2 = (\sigma_a^2 - \sigma_2^2)/\sigma_2^2 \times 100\%$ is given in the 3rd column. Its value is found to always be negative. The salvo kill probability associated with $\sigma_2 = 0$ is denoted as P_0 and is in the fourth column. The fifth column is the upper bound $\bar{P}$ of the maximum salvo kill probability given by (4.6) and (4.7). In the sixth column, we have the true maximum salvo kill probability designated simply as P which is calculated by (2.21), (2.22) with the optimum σ_2^2 given in the 2nd column. The entries in the 7th column are the salvo kill probabilities associated with the approximately optimal dispersion σ_a^2. The notation P_a is used for it. The last column is the relative loss in the kill probability which would be caused if we were to use the approximate value σ_a^2 instead of

Table 7 Accuracy of Approximation (4.10)

A $z = 400$, $\sigma_a^2/\alpha^2 = 8.60$

n	σ_2^2/α^2	$\Delta\sigma_a^2\%$	P_0	$\bar{P}$	P	P_a	$\Delta P\%$
5	9.34	-7.9	0.0225	0.0414	0.0410	0.0410	0.0
0	9.46	-9.1	0.0634	0.1484	0.1456	0.1454	0.1
5	9.57	-10.2	0.1088	0.2777	0.2709	0.2704	0.2
0	9.67	-11.1	0.1556	0.4063	0.3952	0.3943	0.2
5	9.76	-11.9	0.2024	0.5229	0.5081	0.5067	0.3
0	9.84	-12.6	0.2484	0.6230	0.6056	0.6038	0.3
5	9.92	-13.3	0.2931	0.7061	0.6873	0.6851	0.3
0	9.98	-13.9	0.3362	0.7733	0.7542	0.7518	0.3

B $z = 100$, $\sigma_a^2/\alpha^2 = 3.89$

n	σ_2^2/α^2	$\Delta\sigma_a^2\%$	P_0	$\bar{P}$	P	P_a	$\Delta P\%$
4	4.31	-9.9	0.0600	0.0905	0.0891	0.0890	0.1
8	4.41	-11.8	0.1703	0.2947	0.2872	0.2866	0.2
2	4.48	-13.4	0.2828	0.4952	0.4809	0.4794	0.3
6	4.55	-14.6	0.3877	0.6558	0.6371	0.6350	0.3
0	4.61	-15.7	0.4817	0.7727	0.7529	0.7503	0.3
4	4.66	-16.6	0.5641	0.8534	0.8348	0.8321	0.3
8	4.70	-17.4	0.6352	0.9071	0.8910	0.8885	0.3
2	4.74	-18.1	0.6959	0.9419	0.9288	0.9266	0.2

C $z = 25$, $\sigma_a^2/\alpha^2 = 1.53$

n	σ_2^2/α^2	$\Delta\sigma_a^2\%$	P_0	$\bar{P}$	P	P_a	$\Delta P\%$
2	1.77	-13.6	0.0573	0.0660	0.0652	0.0651	0.1
4	1.82	-15.9	0.2137	0.2733	0.2663	0.2657	0.2
6	1.86	-17.8	0.3706	0.4822	0.4674	0.4658	0.3
8	1.90	-19.3	0.5076	0.6497	0.6299	0.6274	0.4
10	1.93	-20.5	0.6206	0.7710	0.7497	0.7468	0.4
12	1.95	-21.5	0.7107	0.8539	0.8339	0.8309	0.4
14	1.98	-22.4	0.7812	0.9085	0.8912	0.8884	0.3
16	1.99	-23.1	0.8356	0.9435	0.9295	0.9271	0.3

the true optimal σ_2^2, $\Delta P = (P - P_a)/P \times 100\%$. It is noted that in the case with a large z value the salvo kill probability is much improved by employing the optimal ballistic dispersion. For instance, when $z = 400$, the ratio P/P_0 is around 2.5. But the improvement is not so much when z gets small. In the case with $z = 25$, the ratio is at most 1.25.

As to our approximation, we observe the following.

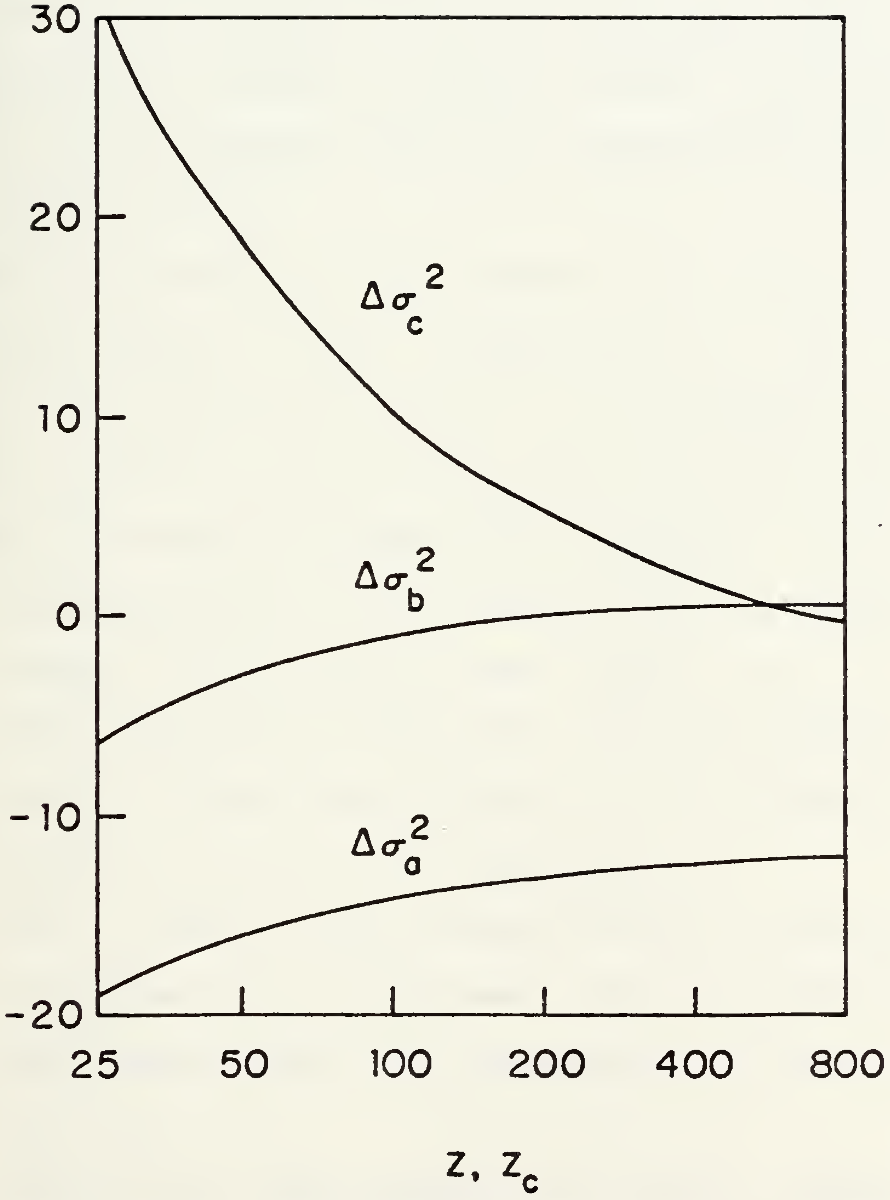
(1) The approximate σ_a^2 is always smaller than the true value, at least for the values of z investigated.

(2) The discrepancy between the approximate and the true values is larger for smaller z.

(3) For a fixed z, the optimal ballistic dispersion σ_2^2 increases as n is increased. Thus the discrepancy grows when n increases for a given z.

(4) However, we may say that the discrepancy is not so serious because the kill probability associated with the approximate optimal ballistic dispersion is not so different from the maximum value. The relative loss given in the last column is at most 0.4% which occurs when z is 25, a less important case.

Figure 5 is presented here to illustrate the properties 1 and 2. The ordinate is the relative error $\Delta\sigma_a^2$, but it needs several words. The relative error $\Delta\sigma_a^2$ varies with n for a fixed z as seen in Table 7. To get a representative $\Delta\sigma_a^2$ value for a z, an n is picked for the given z such that the maximum salvo kill probability is close to 0.6, an



30
20
10
0
-10
-20
$\Delta\sigma^2$ %
$\Delta\sigma_c^2$
$\Delta\sigma_b^2$
$\Delta\sigma_a^2$
25
50
100
200
400
800
Z, Z_c

Figure 5.

arbitrarily chosen figure. Then the corresponding $\Delta\sigma_a^2$ is chosen for that z value, for instance -12.63 for z = 400, and similar procedure follows for other z's. The bottom curve in Figure 5 shows this relationship. Later we will discuss other curves in the same figure.

From the figure we see that the relative error is always negative and its absolute value is decreasing in z. The z-dependence is, however, relatively small, and we learn its bias to the negative side is a characteristic of this approximate formula.

Similar investigation was carried out on the classical approximate formula (4.13). Table 8A for $z_c = 400$, Table 8B for $z_c = 100$, and Table 8C for $z_c = 25$ correspond to Table 7, the only difference being that z_c is tabulated, rather than z. From the tables we observe the following.

(1) The approximate σ_c^2/α^2 is larger than the true value except for the cases of very large z_c's.

(2) The discrepancy between the approximate and the true value is larger for smaller z_c. This tendency is similar to that of our approximation.

(3) For a fixed z_c, the optimal ballistic dispersion σ_2^2 increases as n is increased. Thus, the discrepancy gets smaller when n is increased for a given z_c less than 800.

(4) However, the discrepancy is not serious because the kill probability associated with the approximate

Table 8 Accuracy of Approximation (4.13)

A $z_c = 400, \ \sigma_c^2/\alpha^2 = 10.00$

σ_2^2/α^2	$\Delta\sigma_c^2\%$	P_o	$\bar{P}$	P	P_c	$\Delta P\%$
9.34	7.1	0.0225	0.0414	0.0410	0.0410	0.0
9.46	5.7	0.0634	0.1484	0.1456	0.1455	0.0
9.57	4.5	0.1088	0.2777	0.2709	0.2709	0.0
9.67	3.4	0.1556	0.4063	0.3952	0.3951	0.0
9.76	2.5	0.2024	0.5229	0.5081	0.5080	0.0
9.84	1.6	0.2484	0.6230	0.6056	0.6056	0.0
9.92	0.9	0.2931	0.7061	0.6873	0.6873	0.0
9.98	0.2	0.3362	0.7733	0.7542	0.7542	0.0

B $z_c = 100, \ \sigma_c^2/\alpha^2 = 5.00$

σ_2^2/α^2	$\Delta\sigma_c^2\%$	P_o	$\bar{P}$	P	P_c	$\Delta P\%$
4.31	15.9	0.0600	0.0905	0.0891	0.0889	0.2
4.41	13.5	0.1703	0.2947	0.2872	0.2866	0.2
4.48	11.5	0.2828	0.4952	0.4809	0.4800	0.2
4.55	9.9	0.3877	0.6558	0.6371	0.6363	0.1
4.61	8.5	0.4817	0.7727	0.7529	0.7523	0.1
4.66	7.3	0.5641	0.8534	0.8348	0.8344	0.1
4.70	6.3	0.6352	0.9071	0.8910	0.8907	0.0
4.74	5.4	0.6959	0.9419	0.9288	0.9286	0.0

C $z_c = 25, \ \sigma_c^2/\alpha^2 = 2.50$

σ_2^2/α^2	$\Delta\sigma_c^2\%$	P_o	$\bar{P}$	P	P_c	$\Delta P\%$
1.77	41.0	0.0573	0.0660	0.0652	0.0648	0.5
1.82	37.1	0.2137	0.2733	0.2663	0.2638	0.9
1.86	34.1	0.3706	0.4822	0.4674	0.4631	0.9
1.90	31.7	0.5076	0.6497	0.6299	0.6249	0.8
1.93	29.7	0.6206	0.7710	0.7497	0.7451	0.6
1.95	28.0	0.7107	0.8539	0.8339	0.8300	0.5
1.98	26.5	0.7812	0.9085	0.8912	0.8881	0.3
1.99	25.3	0.8356	0.9435	0.9295	0.9272	0.2

optimal dispersion is not so different from the true maximum. The relative loss is at most 1%, and is very small when σ_1^2/α^2 is large.

The top curve in Figure 5 is the relative error $\Delta\sigma_c^2$ of the classical approximate formula, where the abscissa should be read as z_c in this case. In spite of its simple form, it is indeed an excellent approximation, in particular for large z_c's. The z_c-dependence of the relative error $\Delta\sigma_c^2$ is, however, more sharp than ours. The middle curve in Figure 5 will be discussed in the next section.

B. IMPROVEMENT OF THE APPROXIMATION

In the preceding section, we investigated the accuracy of the approximate formula given by (4.10), and it seemed that (4.10) gave a lower bound to the true optimal ballistic dispersion. The reason it gives a possible lower bound, and a method of obtaining improved approximations will be considered now.

The Walsh optimal $\tilde{f}_2^*$ and others in (3.16) are functions of x,

$$x = (\omega_1^2 + \omega_2^2)\alpha^2/2 , \qquad (4.15)$$

for the Simple Salvo MOdel, and so let us denote them as $\tilde{f}_2^*(x)$ instead of $\tilde{f}_2^*(\omega_1,\omega_2)$, etc. With this notation, (3.18) is

$$\tilde{f}_2^*(x) = \tilde{p}^*(x)/\tilde{D}(x) . \tag{4.16}$$

Equation (3.27) by which the approximate σ_a^2/α^2 is determined reads

$$\tilde{f}_2(x) = e^{-x\sigma_a^2/\alpha^2} = \tilde{p}_0(x)/\tilde{D}_0(x) , \tag{4.17}$$

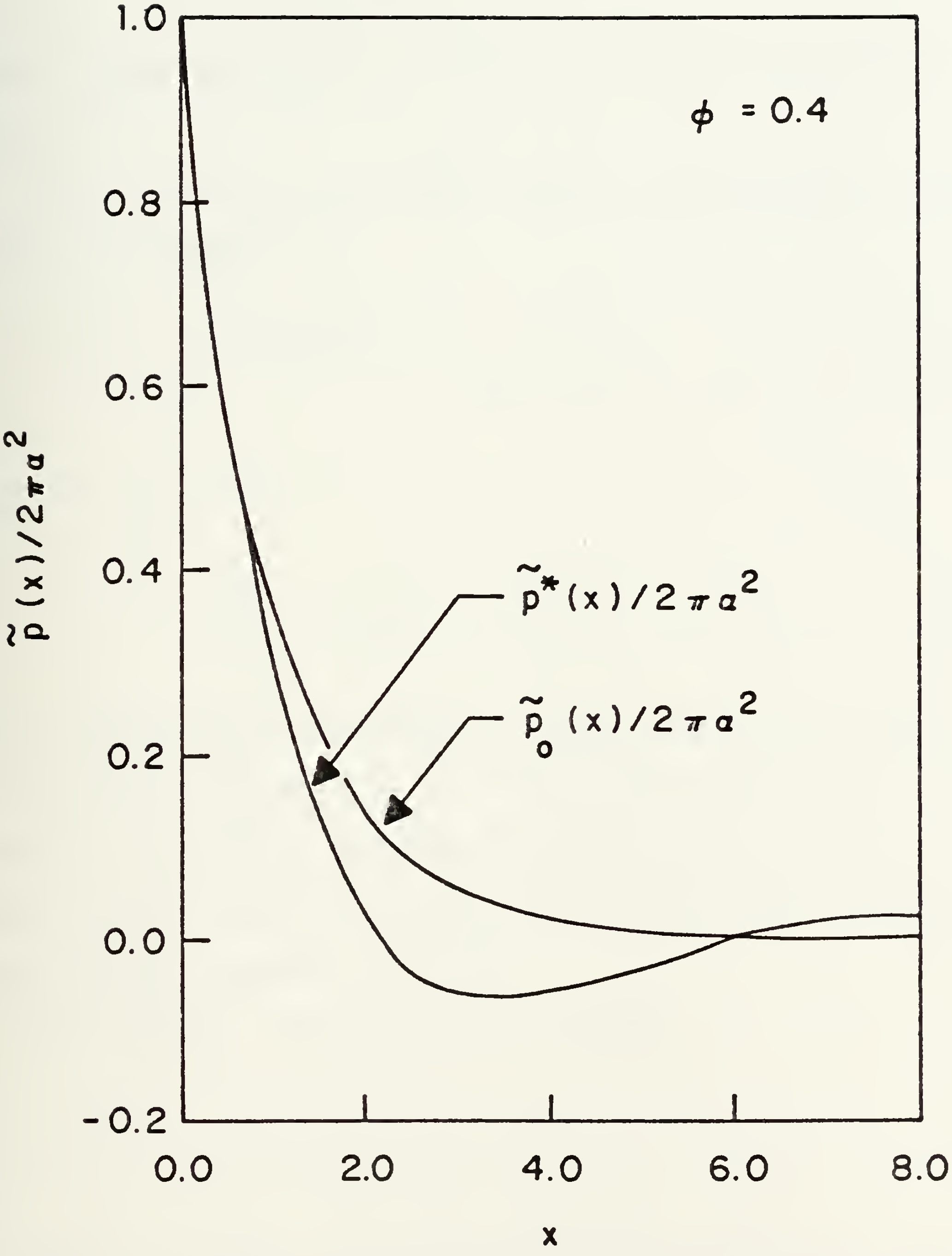
and

$$\tilde{D}(x) = \tilde{D}_0(x) , \tag{4.18}$$

because the damage function is assumed to be Gaussian.

The function $\tilde{p}^*(x)$ is the double Fourier transform of the Walsh optimal p function given in Corollary 3.2 with $\sigma_u = \sigma_v = \sigma_1$, and $\tilde{p}_0(x)$ is an approximate formula for it given by (3.26). Figure 6 illustrates what the curve of $\tilde{p}^*(x)$ is like. The curve decreases as x is increased from zero, changes its sign, and then swings back to positive with small amplitude, and so forth. On the other hand, the approximating function $\tilde{p}_0(x)$ is equal to $\tilde{p}^*(x)$ at $x = 0$, has the same tangent there, and decreases exponentially. $\tilde{p}_0(x)$ is larger than $\tilde{p}^*(x)$ for a wide interval of x as is seen in Figure 6.

$$\tilde{p}_0(x) > \tilde{p}^*(x) , \quad 0 < x < x_0 .$$



$\phi = 0.4$
$\tilde{p}^*(x)/2\pi a^2$
$\tilde{p}_o(x)/2\pi a^2$
$\tilde{p}(x)/2\pi a^2$
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0.0
2.0
4.0
6.0
8.0
x

Figure 6.

Therefore, from (4.16) and (4.17), we have

$$e^{-x\sigma_a^2/\alpha^2} > f_2^*(x) , \quad 0 < x < x_0$$

which suggests that σ_a^2 is smaller than the true optimal value.

One of the simplest ways to get an approximate formula would be to define σ_b^2

$$e^{-\theta x\sigma_b^2/\alpha^2} = \tilde{p}_0(x)/\tilde{D}_0(x) \qquad (4.19)$$

with $0 < \theta < 1$, or

$$\sigma_b^2 = \sigma_a^2/\theta , \quad 0 < \theta < 1 , \qquad (4.20)$$

and to determine the parameter θ empirically to achieve satisfactory agreement between the approximate σ_b^2 and the true σ_2^2. Note that the definition of σ_b^2 corresponds to approximating the $\tilde{p}^*(x)$ function by $\tilde{p}_1(x)$ with an exponent $1/\theta$ times the exponent of $\tilde{p}_0(x)$.

For example, let $1/\theta$ be 1.15. Then,

$$\sigma_b^2 = 1.15\ \sigma_a^2 . \qquad (4.21)$$

In Table 9A, Table 9B, and Table 9C, we present the accuracy investigation of (4.21). Accuracy is found much improved.

Table 9 Accuracy of Approximation (4.21).

A $z = 400$, $\sigma_b^2/\alpha^2 = 9.89$

n	σ_2^2/α^2	$\Delta\sigma_b^2\%$	P_0	$\bar{P}$	P	P_b	$\Delta P\%$
5	9.34	5.9	0.0225	0.0414	0.0410	0.0410	0.0
10	9.46	4.5	0.0634	0.1484	0.1456	0.1455	0.0
15	9.57	3.3	0.1088	0.2777	0.2709	0.2709	0.0
20	9.67	2.3	0.1556	0.4063	0.3952	0.3952	0.0
25	9.76	1.3	0.2024	0.5229	0.5081	0.5081	0.0
30	9.84	0.5	0.2484	0.6230	0.6056	0.6056	0.0
35	9.92	-0.3	0.2931	0.7061	0.6873	0.6873	0.0
40	9.98	-1.0	0.3362	0.7733	0.7542	0.7542	0.0

B $z = 100$, $\sigma_b^2/\alpha^2 = 4.47$

n	σ_2^2/α^2	$\Delta\sigma_b^2\%$	P_0	$\bar{P}$	P	P_b	$\Delta P\%$
4	4.31	3.6	0.0600	0.0905	0.0891	0.0891	0.0
8	4.41	1.4	0.1703	0.2947	0.2872	0.2872	0.0
12	4.48	-0.4	0.2828	0.4952	0.4809	0.4809	0.0
16	4.55	-1.8	0.3877	0.6558	0.6371	0.6371	0.0
20	4.61	-3.1	0.4817	0.7727	0.7529	0.7528	0.0
24	4.66	-4.1	0.5641	0.8534	0.8348	0.8347	0.0
28	4.70	-5.0	0.6352	0.9071	0.8910	0.8908	0.0
32	4.74	-5.8	0.6959	0.9419	0.9288	0.9286	0.0

C $z = 25$, $\sigma_b^2/\alpha^2 = 1.76$

n	σ_2^2/α^2	$\Delta\sigma_b^2\%$	P_0	$\bar{P}$	P	P_b	$\Delta P\%$
2	1.77	-0.6	0.0573	0.0660	0.0652	0.0652	0.0
4	1.82	-3.3	0.2131	0.2733	0.2663	0.2663	0.0
6	1.86	-5.4	0.3706	0.4822	0.4674	0.4673	0.0
8	1.90	-7.1	0.5076	0.6497	0.6299	0.6296	0.1
10	1.93	-8.6	0.6206	0.7710	0.7497	0.7493	0.1
12	1.95	-9.7	0.7107	0.8539	0.8339	0.8333	0.1
14	1.98	-10.8	0.7812	0.9085	0.8912	0.8906	0.1
16	1.99	-11.6	0.8356	0.9435	0.9295	0.9239	0.1

The maximum absolute relative error $\Delta\sigma_b^2$ reduces to 12%, and the relative loss in the salvo kill probability is, at most only 0.07%. The middle curve in Figure 5 is the relative error $\Delta\sigma_b^2$.

V. SALVO MODEL WITH NON GAUSSIAN DAMAGE FUNCTION

In this chapter we deal with a salvo kill model with circular normal errors and circularly symmetric general damage function. Under this general assumption, calculation of the salvo kill probability is very difficult, and time-consuming computation is needed for it. Here we propose an approximate method for calculating the salvo kill probability. By this method computation is very easy and yet fairly high accuracy is expected.

Assumptions in this chapter are as follows: A salvo of n weapons is fired. The impact point error is composed of two parts, the random bias common to all n weapons and the round-to-round error. We adopt a Cartesian coordinate system (x,y) the origin of which coincides with the center of impact of the n weapons. The random position of the target is denoted as (U,V) with respect to this coordinate system, and (U,V) is assumed to be circular normal centered at (0,0) with variance σ_1^2:

$$f_1(u,v) = \frac{1}{2\pi\sigma_1^2} e^{-\frac{u^2+v^2}{2\sigma_1^2}} . \tag{5.1}$$

The impact points of the ith weapon (X_i,Y_i) are independent and identically distributed circular normal random variables

centered at (0,0) with variance σ_2^2:

$$f_2(x,y) = \frac{1}{2\pi\sigma_2^2} e^{-\frac{x^2+y^2}{2\sigma_2^2}} . \qquad (5.2)$$

The conditional kill probability of the target by a single weapon given that the target is at (u,v) and the weapon impact point is (x,y) is called as the damage function D. In this chapter, D is assumed to be a function of the miss distance $r = \sqrt{(u-x)^2 + (v-y)^2}$ only.

$$D = D(r) , \quad r^2 = (u-x)^2 + (v-y)^2 . \qquad (5.3)$$

Further, the lethal area is denoted as A, and it is assumed that there exist finite "moments" of all degree as defined in (5.5),

$$2\pi \int_0^\infty D(r)\, r\, dr = A , \qquad (5.4)$$

$$E(r^j) = \frac{2\pi}{A} \int_0^\infty r^j D(r)\, r\, dr < \infty , \quad j = 1,2,... \qquad (5.5)$$

The conditional kill probability by a single weapon given (U,V) being (u,v) is given by (1.1), repeated here as

$$p(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(u-x,v-y) f_2(x,y)\, dx\, dy , \qquad (5.6)$$

and the salvo kill probability (1.2) is renamed as (5.7).

$$P = 1 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1 - p(u,v)\}^n f_1(u,v)\, du\, dv \ . \qquad (5.7)$$

In the Simple Salvo Model in Chapter II with Gaussian assumption on the damage function, we have a simple expression for p(u,v):

$$p(u,v) = \frac{\alpha^2}{\alpha^2 + \sigma_2^2} e^{-\frac{u^2+v^2}{2(\alpha^2 + \sigma_2^2)}} \ . \qquad (2.6)$$

In the general model here, p(u,v) given by (5.6) is a complicated function in $\sqrt{u^2 + v^2}$, and computation of the kill probability is difficult. To overcome this difficulty, we propose an approximate formula for p(u,v), and then present a recursive method for computing the salvo kill probability with this approximate p(u,v) function.

A. APPROXIMATE FORMULA FOR p(u,v)

In this section, an approximate formula for the conditional kill probability p(u,v) is presented. As a corollary, we will get an approximation to the so-called Circular Coverage Function.

Define the double Fourier transform of p(u,v) as

$$\tilde{p}(\omega_1,\omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\omega_1 u+\omega_2 v)} p(u,v)\, du\, dv \ , \qquad (5.8)$$

with similar notations being used for D(x,y) and f(x,y). Then, from (5.6), we have

$$\tilde{p}(\omega_1,\omega_2) = \tilde{D}(\omega_1,\omega_2)\tilde{f}_2(\omega_1,\omega_2) . \tag{5.9}$$

The Fourier transform of the circular normal density (5.2) is

$$\tilde{f}_2(\omega_1,\omega_2) = e^{-\frac{\sigma_2^2}{2}(\omega_1^2+\omega_2^2)} . \tag{5.10}$$

In the Fourier transform of the damage function, let the exponential function be expanded and integrated term by term,

$$\tilde{D}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\omega_1 x+\omega_2 y)} D(x,y)\,dx\,dy$$

$$= \sum_{j=0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{j!}\, i^j(\omega_1 x+\omega_2 y)^j D(x,y)\,dx\,dy .$$

By symmetry, the odd terms are zero, so

$$\tilde{D}(\omega_1,\omega_2) = A\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\frac{1}{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\omega_1 x+\omega_2 y)^{2k} D(x,y)\,dx\,dy$$

$$= A\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\frac{1}{A}\int_0^{\infty}\{\int_0^{2\pi}(\omega_1\cos\theta+\omega_2\sin\theta)^{2k}d\theta\}r^{2k}D(r)\,r\,dr$$

Therefore,

$$\tilde{D}(\omega_1,\omega_2)/A = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \frac{(2k-1)(2k-3)\cdots 1}{2k(2k-2)\cdots 2} E(r^{2k})(\omega_1^2+\omega_2^2)^k \tag{5.11}$$

since

$$\frac{1}{2\pi}\int_0^{2\pi}(\omega_1\cos\theta+\omega_2\sin\theta)^{2k}d\theta = \begin{cases} \dfrac{(2k-1)(2k-3)\cdots 1}{2k(2k-2)\cdots 2}, & k \geqq 1 \\ 1 & k = 0 \end{cases}$$

The first few terms in (5.11) are

$$\tilde{D}(\omega_1,\omega_2)/A = 1 - \frac{1}{4}E(r^2)(\omega_1^2+\omega_2^2) + \frac{1}{64}E(r^4)(\omega_1^2+\omega_2^2)^2 - \ldots$$

Now let us define

$$\tilde{D}_a(\omega_1,\omega_2) = A\{1 - \frac{b}{2}(\omega_1^2+\omega_2^2)\}e^{-c(\omega_1^2+\omega_2^2)} \tag{5.12}$$

and determine two constants b and c such that (5.11) and (5.12) coincide up to the fourth power of ω_i's. Then we get

$$b = \frac{1}{2}\sqrt{\{E(r^2)\}^2 - E(r^4)/2}\ , \tag{5.13}$$

and

$$c = -b/2 + E(r^2)/4 .$$

If (5.10) and (5.13) are substituted in (5.9), the result is

$$\tilde{p}_a(\omega_1,\omega_2) = A\{1 - \frac{b}{2}(\omega_1^2 + \omega_2^2)\} e^{-\frac{S^2}{2}(\omega_1^2+\omega_2^2)} , \qquad (5.14)$$

where

$$S^2 = \sigma_2^2 - b + E(r^2)/2 \qquad (5.15)$$

with b given by (5.13).

Next, the inverse transform of $\tilde{p}_a(\omega_1,\omega_2)$ is calculated:

$$p_a(u,v) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(u\omega_1+v\omega_2)} \tilde{p}_a(\omega_1,\omega_2)\, d\omega_1 d\omega_2 . \qquad (5.16)$$

The inverse transform of the first term in (5.14)

$$Ae^{-S^2(\omega_1^2+\omega_2^2)/2}$$

is easily calculated as

$$\frac{A}{2\pi S^2} e^{-\frac{u^2+v^2}{2S^2}} .$$

The second term in (5.14) is

$$-\frac{Ab}{2}(\omega_1^2+\omega_2^2)e^{-\frac{S^2}{2}(\omega_1^2+\omega_2^2)} = -b\frac{d}{d(S^2)}\{Ae^{-\frac{S^2}{2}(\omega_1^2+\omega_2^2)}\},$$

and its inverse transform is given by

$$-b\frac{d}{d(S^2)}\{\frac{A}{2\pi S^2}e^{-\frac{u^2+v^2}{2S^2}}\}.$$

Therefore we have

$$p_a(u,v) = \frac{A}{2\pi S^2}\{1-\frac{b}{S^2}(1-\frac{u^2+v^2}{2S^2})\}e^{-\frac{u^2+v^2}{2S^2}}. \qquad (5.17)$$

It is noted that (5.17) is exact when the damage function is Gaussian with parameter α; (5.17) coincides with the exact formula (2.6) because in this case $A = 2\pi\alpha^2$, $b = 0$, and $S^2 = \sigma_2^2 + \alpha^2$.

The cookie cutter damage function differs substantially from the Gaussian, so consideration of the former provides a good test of the robustness of (5.17). The cookie cutter damage function is

$$D(r) = \begin{cases} 1 & \text{if } 0 \leqq r \leqq a \\ 0 & \text{otherwise} \end{cases} \qquad (5.18)$$

The probability

$$p(u,v) \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D(u-x,v-y)\,f_2(x,y)\,dx\,dy$$

$$= \frac{1}{2\pi\sigma_2} \iint_{x^2+y^2 \leqq a^2} e^{-\frac{(x-u)^2+(y-v)^2}{2\sigma_2^2}}\,dx\,dy$$

is called the Circular Coverage Function. In the literature, the notation $P(a/\sigma_2,r/\sigma_2)$ is used for it, where $r = \sqrt{u^2+v^2}$. A standard method of calculating the Circular Coverage Function is the algorithm given by Brennan and Reed [Ref. 10] based on an infinite series expansion, but if the accuracy required is not too high, it is convenient to use closed form approximations. Standard handbooks [Refs. 11,12] recommend the next formula for $a/\sigma_2 < 1$.

$$P_A(a/\sigma_2,r/\sigma_2) \;=\; \frac{1}{2\Sigma^2}\,e^{-\frac{r^2}{2\Sigma^2}}\,, \tag{5.19}$$

with

$$\Sigma^2 \;=\; \sigma_2^2 + a^2/4\ . \tag{5.20}$$

According to (5.17), the Circular Coverage Function

$$P(a/\sigma_2,r/\sigma_2) = \frac{1}{2\pi\sigma_2} \iint_{x^2+y^2 \leqq a^2} e^{-\frac{(x-r)^2+y^2}{2\sigma_2^2}}\,dx\,dy \tag{5.21}$$

can be approximated by

$$P_a(a/\sigma_2, r/\sigma_2) = \frac{a^2}{2S^2}\{1 - \frac{b}{S^2}(1 - \frac{r^2}{2S^2})\}e^{-\frac{r^2}{2S^2}} , \qquad (5.22)$$

where

$$b = a^2/4\sqrt{3} , \quad S^2 = \sigma_2^2 + a^2(1 - 1/\sqrt{3})/4 , \qquad (5.23)$$

since $A = \pi a^2$, $E(r^2) = a^2/2$, and $E(r^4) = a^4/3$ for the cookie cutter damage function.

In Table 10, the Circular Coverage Functio $P(R,d)$ is tabulated for $R = 0.1\ (0.1)\ 2.0$ and $d = 0.0\ (0.5)\ 2.5$. In Table 11, the error $\Delta_A(R,d) = P_A(R,d) - P(R,d)$ is shown. For small R, the error is very small; for instance the error is less than 10^{-2} when $R \leqq 1.0$. But when R is increased beyond 1, the error grows rapidly.

In Table 12, the error $\Delta_a(R,d) = P_a(R,d) - P(R,d)$ of our approximate formula is given. The accuracy is found much better than the above-mentioned formula. The absolute error for $d = 0$ is less than 5×10^{-4} when $R \leqq 1.0$, less than 10^{-2} even when $R = 1.6$ and usually shows much higher accuracy.

B. RECURRENCE FORMULA FOR THE SALVO KILL PROBABILITY

If the conditional kill probability $p(u,v)$ by a single weapon is given in the form (5.17), we can easily compute the

Table 10 Circular Coverage Function P(R,d)

	d					
R	0.00	0.50	1.00	1.50	2.00	2.50
0.1	0.0050	0.0044	0.0030	0.0016	0.0007	0.0002
0.2	0.0198	0.0175	0.0121	0.0065	0.0027	0.0009
0.3	0.0440	0.0389	0.0270	0.0146	0.0062	0.0021
0.4	0.0769	0.0682	0.0476	0.0261	0.0112	0.0038
0.5	0.1175	0.1045	0.0735	0.0408	0.0179	0.0062
0.6	0.1647	0.1470	0.1043	0.0589	0.0264	0.0094
0.7	0.2173	0.1946	0.1397	0.0803	0.0369	0.0135
0.8	0.2739	0.2463	0.1790	0.1050	0.0495	0.0187
0.9	0.3330	0.3009	0.2217	0.1328	0.0645	0.0252
1.0	0.3935	0.3573	0.2671	0.1638	0.0819	0.0332
1.1	0.4539	0.4144	0.3146	0.1976	0.1019	0.0428
1.2	0.5132	0.4712	0.3635	0.2341	0.1247	0.0543
1.3	0.5704	0.5267	0.4132	0.2730	0.1501	0.0679
1.4	0.6247	0.5802	0.4628	0.3138	0.1784	0.0838
1.5	0.6753	0.6309	0.5120	0.3563	0.2092	0.1021
1.6	0.7220	0.6785	0.5599	0.3999	0.2426	0.1228
1.7	0.7643	0.7224	0.6062	0.4442	0.2784	0.1462
1.8	0.8021	0.7625	0.6504	0.4887	0.3161	0.1723
1.9	0.8355	0.7987	0.6921	0.5329	0.3556	0.2010
2.0	0.8647	0.8309	0.7310	0.5763	0.3965	0.2321

Table 11 Error $\Delta_A(R,d)$

	d					
R	0.00	0.50	1.00	1.50	2.00	2.50
0.1	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.2	0.0000	0.0000	0.0000	-0.0000	-0.0000	-0.0000
0.3	0.0000	0.0000	0.0000	-0.0000	-0.0000	-0.0000
0.4	0.0000	0.0000	0.0000	-0.0000	-0.0000	-0.0000
0.5	0.0001	0.0001	0.0000	-0.0000	-0.0000	-0.0000
0.6	0.0004	0.0003	0.0000	-0.0001	-0.0001	-0.0000
0.7	0.0010	0.0007	0.0001	-0.0002	-0.0002	-0.0000
0.8	0.0020	0.0014	0.0003	-0.0004	-0.0003	-0.0001
0.9	0.0038	0.0027	0.0005	-0.0007	-0.0006	-0.0002
1.0	0.0065	0.0046	0.0010	-0.0012	-0.0011	-0.0003
1.1	0.0106	0.0076	0.0018	-0.0018	-0.0019	-0.0007
1.2	0.0162	0.0118	0.0030	-0.0026	-0.0030	-0.0011
1.3	0.0236	0.0174	0.0048	-0.0036	-0.0045	-0.0019
1.4	0.0330	0.0246	0.0074	-0.0047	-0.0065	-0.0030
1.5	0.0447	0.0337	0.0109	-0.0058	-0.0090	-0.0046
1.6	0.0585	0.0447	0.0155	-0.0068	-0.0121	-0.0067
1.7	0.0746	0.0578	0.0213	-0.0076	-0.0157	-0.0095
1.8	0.0929	0.0728	0.0286	-0.0080	-0.0197	-0.0131
1.9	0.1132	0.0898	0.0374	-0.0077	-0.0240	-0.0174
2.0	0.1353	0.1086	0.0478	-0.0065	-0.0286	-0.0225

Table 12 Error $\Delta_q(R,d)$

	d					
R	0.00	0.50	1.00	1.50	2.00	2.50
0.1	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.2	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.3	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.4	-0.0000	-0.0000	-0.0000	0.0000	-0.0000	-0.0000
0.5	-0.0000	-0.0000	0.0000	0.0000	0.0000	-0.0000
0.6	-0.0000	-0.0000	0.0000	0.0000	0.0000	-0.0000
0.7	-0.0000	-0.0000	0.0000	0.0000	0.0000	-0.0000
0.8	-0.0001	-0.0000	0.0000	0.0000	0.0000	-0.0000
0.9	-0.0001	-0.0001	0.0000	0.0000	0.0000	-0.0000
1.0	-0.0003	-0.0002	0.0000	0.0001	0.0000	-0.0000
1.1	-0.0006	-0.0003	0.0000	0.0001	0.0000	-0.0000
1.2	-0.0011	-0.0007	0.0000	0.0003	0.0001	-0.0000
1.3	-0.0019	-0.0012	0.0000	0.0005	0.0002	-0.0001
1.4	-0.0031	-0.0019	0.0000	0.0008	0.0003	-0.0001
1.5	-0.0049	-0.0031	-0.0000	0.0012	0.0006	-0.0001
1.6	-0.0075	-0.0048	-0.0001	0.0019	0.0009	-0.0002
1.7	-0.0110	-0.0072	-0.0003	0.0027	0.0015	-0.0002
1.8	-0.0157	-0.0104	-0.0007	0.0039	0.0023	-0.0002
1.9	-0.0218	-0.0146	-0.0012	0.0053	0.0034	-0.0001
2.0	-0.0294	-0.0199	-0.0021	0.0071	0.0049	0.0001

salvo kill probability in a recursive way. We first observe that the salvo kill probability

$$P = 1 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1 - p(u,v)\}^n f_1(u,v)\, du\, dv$$

where $f_1(u,v)$ is the circular normal density (5.1) and

$$p(u,v) = (\mu + \beta \frac{r^2}{2S^2})\, e^{-\frac{r^2}{2S^2}} , \quad r^2 = u^2 + v^2 , \tag{5.24}$$

can be expressed as

$$P = 1 - K(n,0) , \tag{5.25}$$

where

$$K(n,0) = \int_0^{\infty} \{1 - (\mu + \beta t/\rho) e^{-t/\rho}\}^n e^{-t}\, dt , \tag{5.26}$$

and where $\rho = S^2/\sigma_1^2$. This is simply a matter of substitution and the introduction of polar coordinates. We then have

Proposition 5.1

The function

$$K(n,j) = \int_0^{\infty} \{1 - (\mu + \beta t/\rho) e^{-t/\rho}\}^n e^{-(j/\rho+1)t} dt \tag{5.27}$$

which has two integer arguments and three parameters μ, β, and ρ satisfies the following recurrence relation:

$$K(n,j) = \frac{n}{n+j+\rho} K(n-1,j) + \frac{\rho}{n+j+\rho}(1-\mu)^n - \frac{n\beta}{n+j+\rho} K(n-1,j+1) \ ,$$

$$n = 1,2, \ldots ; \quad j = 0,1,\ldots \qquad (5.28)$$

with

$$K(0,j) = \frac{\rho}{j+\rho} \ , \quad j = 0,1, \ldots \qquad (5.29)$$

Proof

From (5.27),

$$K(n,j) = \int_0^\infty \{1 - (\mu + \beta t/\rho)e^{-t/\rho}\}^{n-1} e^{-(j/\rho+1)t} \, dt$$

$$- \int_0^\infty \{1 - (\mu + \beta t/\rho)e^{-t/\rho}\}^{n-1} (\mu + \beta t/\rho)e^{-t/\rho} e^{-(j/\rho+1)t} dt$$

The first term in the right-hand-side is $K(n-1,j)$. The second terms is, when integrated by parts,

$$- \int_0^\infty \{1 - (\mu + \beta t/\rho) e^{-t/\rho}\}^{n-1} (\mu + \beta t/\rho) e^{-t/\rho} e^{-(j/\rho+1)t} dt$$

$$= - \frac{\rho}{n}\{1 - (\mu + \beta t/\rho) e^{-t/\rho}\}^n e^{-(j/\rho+1)t} \Big|_0^\infty$$

$$- \frac{\rho}{n}(j/\rho+1) \int_0^\infty \{1 - (\mu + \beta t/\rho) e^{-t/\rho}\}^n e^{-(j/\rho+1)t} \, dt$$

$$- \beta \int_0^\infty \{1 - (\mu + \beta t/\rho) e^{-t/\rho}\}^n e^{-t/\rho} e^{-(j/\rho+1)t} \, dt$$

$$= \frac{\rho}{n}(1-\mu)^n - \frac{j+\rho}{n} K(n,j) - \beta K(n-1,j+1) \; .$$

Therefore

$$K(n,j) = K(n-1,j) + \frac{\rho}{n}(1-\mu)^n - \frac{j+\rho}{n} K(n,j) - \beta K(n-1,j+1)$$

from which (5.28) is immediate. From the definition of $K(0,j)$, it is clear that (5.29) holds.

Note 1

We use the notations μ and ρ here in the same way as in Chapter II without any subscripts because no confusion is expected. Their definitions are different, but they are just the counterparts of μ and ρ defined in Chapter II.

Note 2

It is worthwhile to note that the recurrence relation (5.28) has a similar form to (2.17). The p function (5.24)

involves an extra parameter β which was missing in the Simple Salvo Model in Chapter II, and therefore, it is necessary to introduce another index j.

VI. COOKIE CUTTER SALVO MODEL

Consider that n weapons are fired at a target in a salvo. The impact point error is composed of two parts, the random bais and the dispersion. We adopt a Cartesian coordinate system (x,y) the origin of which coincides with the center of impact of the n weapons. The random position of the target is denoted as (U,V) with respect to this coordinate system, and (U,V) is assumed to be circular normal centered at (0,0) with variance σ_1^2.

$$f_1(u,v) = \frac{1}{2\pi\sigma_1^2} e^{-\frac{u^2+v^2}{2\sigma_1^2}} . \tag{6.1}$$

The impact points of the ith weapon (X_i,Y_i) are independent and identically distributed circular normal random variables centered at (0,0) with variance σ_2^2.

$$f_2(x,y) = \frac{1}{2\pi\sigma_2^2} e^{-\frac{x^2+y^2}{2\sigma_2^2}} . \tag{6.2}$$

The kill probability by a single weapon conditional on (U,V) = (u,v) and (X,Y) = (x,y) is a function of u-x and v-y, D(u-x,v-y), and in this chapter it is assumed that

$$D(u-x,v-y) = \begin{cases} 1 & \text{if } (u-x)^2 + (v-y)^2 \leqq a^2 \\ 0 & \text{otherwise ,} \end{cases} \qquad (6.3)$$

with lethal area

$$A = \pi a^2 .$$

In the following, we call this model the Cookie Cutter Salvo Model.

The conditional probability that the target is destroyed by a single weapon given the random position of the target being (u,v) is

$$p(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(u-x,v-y) f_2(x,y)\, dx\, dy ,$$

and with (6.2) and (6.3), we have

$$p(u,v) = \frac{1}{2\pi\sigma_2^2} \iint_{x^2+y^2 \leqq a^2} e^{-\frac{(u-x)^2+(v-y)^2}{2\sigma_2^2}} dx dy = P(a/\sigma_2, \sqrt{u^2+v^2}/\sigma_2) , \qquad (6.4)$$

where P(R,d) is the Circular Coverage Function.

The salvo kill probability

$$P_c = 1 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1 - p(u,v)\}^n f_1(u,v)\, du\, dv ,$$

with (6.1) and (6.4) is readily rewritten as

$$P_c = 1 - \int_0^{\infty} \{1 - P(a/\sigma_2, \sqrt{2t}\sigma_1/\sigma_2)\}^n e^{-t}\, dt \; . \qquad (6.5)$$

We carried out a number of computations of (6.5) with various sets of parameters σ_1, σ_2, α, and n by the Simpson method. A robust algorithm for computing the Circular Coverage Function P(R,d) for wide ranges of R and d was needed. For this purpose, we used the Brennan-Reed formula in a modified form.

Brennan-Reed Formula:

$$P(R,d) = \sum_{n=0}^{\infty} g_n k_n \; ,$$

$$k_n/k_{n-1} = d^2/(2n) \; , \quad n = 1,2, \ldots$$

$$k_0 = e^{-d^2/2} \; ,$$

$$g_n = g_{n-1} - (R^2/2)^n \, e^{-R^2/2}/n! \; , \quad n = 1,2, \ldots$$

$$g_0 = 1 - e^{-R^2/2} \; .$$

Modified Formula:

$$P(R,d) = e^{-d^2/2} \sum_{n=0}^{\infty} (1 - e^{-R^2/2} C_n) h_n , \qquad (6.7)$$

$$h_n/h_{n-1} = d^2/(2n) , \quad n = 1,2, \ldots$$

$$h_0 = 1 ,$$

$$C_n = C_{n-1} + a_n , \quad n = 1,2, \ldots \qquad (6.8)$$

$$a_n/a_{n-1} = R^2/(2n) , \quad n = 1,2, \ldots$$

$$C_0 = a_0 = 1 .$$

The modified formula is especially of use in preventing underflow possible in cases with large R and d.

In their review paper, Eckler and Burr presented a short table of the salvo kill probability of the Cookie Cutter Salvo Model. But the accuracy is, as they admitted, supposedly low because it is obtained by Monte Carlo simulation. In Table 13, we give the correct values. The figures in parentheses are from the table by Eckler and Burr.

In the following sections, comparison of the Cookie Cutter Salvo Model and the Simple Salvo Model is dealt with and then the optimal ballistic dispersion is investigated, where

Table 13 Salvo Kill Probability : Kisi/(Eckler-Burr)

		σ_2/σ_1		
	a/σ_1	0.25	0.50	0.75
	0.6	0.221	0.228	0.199
		(0.224)	(0.235)	(0.201)
	1.2	0.573	0.587	0.557
		(0.574)	(0.582)	(0.563)
A n = 2	1.8	0.834	0.836	0.813
		(0.834)	(0.832)	(0.814)
	2.4	0.954	0.952	0.939
		(0.949)	(0.951)	(0.942)
	3.0	0.991	0.990	0.985
		(0.990)	(0.988)	(0.985)

		σ_2/σ_1		
	a/σ_1	0.50	0.75	1.00
	0.4	0.231	0.213	0.177
		(0.225)	(0.214)	(0.187)
	0.8	0.525	0.551	0.514
		(0.521)	(0.551)	(0.516)
B n = 5	1.2	0.736	0.772	0.764
		(0.728)	(0.773)	(0.763)
	1.6	0.872	0.894	0.896
		(0.865)	(0.892)	(0.893)
	2.0	0.946	0.957	0.958
		(0.945)	(0.956)	(0.957)

			σ_2/σ_1		
	a/σ_1	0.50	0.75	1.00	1.50
	0.2	0.221	0.210	0.176	0.115
		(0.209)	(0.201)	(0.176)	(0.113)
	0.4	0.458	0.525	0.504	0.381
		(0.447)	(0.531)	(0.499)	(0.386)
	0.6	0.605	0.713	0.740	0.648
		(0.595)	(0.716)	(0.740)	(0.640)
C n = 20	0.8	0.715	0.818	0.861	0.828
		(0.706)	(0.817)	(0.846)	(0.828)
	1.0	0.801	0.884	0.922	0.923
		(0.792)	(0.882)	(0.924)	(0.924)
	1.2	0.865	0.928	0.956	0.966
		(0.856)	(0.926)	(0.958)	(0.970)
	1.4	0.912	0.956	0.975	0.985
		(0.900)	(0.953)	(0.975)	(0.984)

the approximate formula presented in Chapter III and the approximate method of computing the salvo kill probability developed in Chapter V are found to be useful.

A. COOKIE CUTTER VS. SIMPLE SALVO MODEL

The Cookie Cutter Salvo Model differs from the Simple Salvo Model of Chapter II only in the damage function. In Chapter II, we assumed

$$D(u-x,v-y) = e^{-\frac{(u-x)^2+(v-y)^2}{2\alpha^2}} ,$$

with lethal area $2\pi\alpha^2$. For comparison's sake, the lethal area of the two salvo models are now set equal

$$a^2 = 2\alpha^2 . \tag{6.9}$$

The two models share the common parameters σ_1, σ_2, α, and n, only being different with respect to the shape of the damage function. The quantities associated with the Cookie Cutter Salvo Model will be given suffix c in the following when necessary.

It is interesting to compare the salvo kill probabilities of the two models. We begin with three extreme cases in which the salvo kill probability has closed form expressions.

(1) The case $n = 1$.

The formula (6.5) with (6.4) is easily integrated and give

$$P_c = 1 - e^{-\alpha^2/(\sigma_1^2+\sigma_2^2)} \quad . \tag{6.10}$$

The corresponding single shot kill probability under the assumption of the Gaussian damage function is

$$P = 1 - \frac{1}{1+\alpha^2/(\sigma_1^2+\sigma_2^2)} \quad . \tag{6.11}$$

Proposition 6.1

The Cookie Cutter Salvo Model with n = 1 gives a higher single shot kill probability than that of the Simple Salvo Model.

$$P_c > P \ . \tag{6.12}$$

Proof

Recall that $e^{-x} < 1/(1+x)$ for $x > 0$. Let $\alpha^2/(\sigma_1^2+\sigma_2^2) = x$ in (6.10) and (6.11), then (6.12) is obvious. □

Generally, in the salvo with $n > 1$, this inequality does not necessarily hold as will be shown in the following.

(2) The case $\sigma_1 = 0$

Let $\sigma_1 \rightarrow 0$ in (6.5), then we have

$$P_c = 1 - e^{-n\alpha^2/\sigma_2^2} \quad . \tag{6.13}$$

The corresponding formula in the Simple Salvo Model is given by(2.23), namely

$$P = 1 - 1/(1+\alpha^2/\sigma_2^2)^n . \tag{6.14}$$

(3) The case $\sigma_2 = 0$

Let $\sigma_2 \to 0$ in (6.4). We get

$$P_c = 1 - e^{-\alpha^2/\sigma_1^2} . \tag{6.15}$$

The corresponding formula in the Simple Salvo Model is given by (2.24):

$$P = 1 - \frac{n(n-1)\cdots 1}{(n+\lambda)(n-1+\lambda)\cdots(1+\lambda)} , \tag{6.16}$$

with

$$\lambda = \alpha^2/\sigma_1^2 .$$

Proposition 6.2

Between the salvo kill probabilities of the Cookie Cutter Salvo Model and the Simple Salvo Model with the same parameters, the following relationships hold:

If $\sigma_1 = 0$, then

$$P_c > P . \tag{6.17}$$

If $\sigma_2 = 0$, then

$$P_c \gtrless P \qquad \text{iff} \quad \sigma_1^2/\alpha^2 \lesseqgtr 1/\lambda_0 \ , \tag{6.18}$$

where λ_0 is a positive solution to the equation

$$\lambda - \sum_{j=1}^{n} \ln(1 + \lambda/j) = 0 \ . \tag{6.19}$$

Proof

The proof of (6.17) is just the same as that of Proposition 6.1. As to the second assertion, let

$$P = 1 - e^{-g(\lambda)}$$

in (6.16). Then

$$g(\lambda) = \sum_{j=1}^{n} \ln(1 + \lambda/j) \ , \qquad dg/d\lambda = \sum_{j=1}^{n} 1/(j+\lambda) \ .$$

Since $g(0) = 0$, $dg/d\lambda|_{\lambda=0} > 1$, $d^2g/d\lambda^2 < 0$, we have

$$g(\lambda) \gtrless \lambda \qquad \text{iff} \quad \lambda \lesseqgtr \lambda_0 \ .$$

where λ_0 is given by (6.19). The relation (6.18) is immediate. □

As to the general case, we must compare the two kill probabilities numerically. To illustrate the general idea, Table 14 is given. In Table 14A, P_c for $n = 4$ is tabulated, and the corresponding Simple Salvo Model kill probability P is in Table 14B. It is observed that P_c's in the column $\sigma_1^2/\sigma^2 = 0$ are larger than P's as stated in Proposition 6.2. P_c's in the first row corresponding to $\sigma_2^2/\alpha^2 = 0$ is smaller than P's except for the case $\sigma_1^2/\alpha^2 = 0$, since $1/\lambda_0 = 0.216$ for $n = 4$.

As a general tendency, P_c is larger than P when σ_2 is relatively larger than σ_1. P_c is smaller than P when σ_2 is considerably smaller than σ_1, as is seen in the upper right corner of the table.

B. OPTIMAL BALLISTIC DISPERSION

We carried out a golden search calculation to get the optimal ratio σ_2^2/α^2 for a number of cases with different combinations of n and σ_1^2/α^2. The results are tabulated in Table 15. Table 15A gives the optimal σ_{2c}^2/α^2 of the Cookie Cutter Salvo Model, and Table 15B gives the associated salvo kill probability, namely the maximum salvo kill probability for a given set of n and σ_1^2/α^2. It is noted that this calculation is very time-consuming. The corresponding values for the Simple Salvo Model are tabulated in Table 16.

Before starting comparison of the optimal ballistic dispersion of the Cookie Cutter Salvo Model with that of the

Table 14 Salvo Kill Probability, n = 4

A Cookie Cutter Salvo Model, P_c.

	σ_1^2/α^2					
σ_2^2/α^2	0.0	2.0	4.0	6.0	8.0	10.0
0.0	1.000	0.393	0.221	0.154	0.118	0.095
0.5	1.000	0.605	0.389	0.285	0.224	0.185
1.0	0.982	0.626	0.425	0.320	0.256	0.214
1.5	0.931	0.615	0.435	0.335	0.272	0.229
2.0	0.865	0.592	0.432	0.339	0.279	0.237
2.5	0.798	0.565	0.423	0.337	0.281	0.240
3.0	0.736	0.538	0.412	0.333	0.279	0.241
3.5	0.681	0.511	0.399	0.326	0.276	0.240
4.0	0.632	0.486	0.385	0.319	0.272	0.238

B Simple Salvo Model, P.

	σ_1^2/α^2					
σ_2^2/α^2	0.0	2.0	4.0	6.0	8.0	10.0
0.0	1.000	0.594	0.382	0.280	0.221	0.183
0.5	0.988	0.621	0.421	0.317	0.253	0.211
1.0	0.938	0.613	0.431	0.331	0.268	0.225
1.5	0.870	0.589	0.427	0.334	0.274	0.232
2.0	0.802	0.560	0.417	0.331	0.274	0.233
2.5	0.740	0.531	0.404	0.325	0.271	0.233
3.0	0.684	0.503	0.390	0.317	0.267	0.230
3.5	0.634	0.476	0.375	0.309	0.262	0.227
4.0	0.590	0.452	0.361	0.300	0.256	0.223

Table 15 Optimal Ballistic Dispersion and Maximum Probability (Cookie Cutter Salvo Model)

A Optimal Ratio σ_{2c}^2 / α^2 .

n

σ_1^2/α^2	2	4	8	16	32	64
2	0.4	1.0	1.7	2.7	4.1	6.1
4	0.8	1.6	2.6	3.9	5.8	8.6
6	1.2	2.1	3.2	4.9	7.1	10.5
8	1.6	2.5	3.8	5.6	8.3	12.0
10	1.8	2.9	4.4	6.4	9.3	13.4
12	2.1	3.3	4.9	7.0	10.1	14.7

B Maximum Salvo Kill Probability

$P_{c\,max}$

n

σ_1^2/α^2	2	4	8	16	32	64
2	0.476	0.626	0.782	0.905	0.973	0.996
4	0.300	0.435	0.601	0.769	0.899	0.971
6	0.224	0.339	0.492	0.667	0.826	0.935
8	0.182	0.281	0.420	0.590	0.762	0.897
10	0.154	0.241	0.368	0.531	0.708	0.859
12	0.135	0.213	0.329	0.484	0.661	0.823

Table 16 Optimal Ballistic Dispersion and Maxmum Probability (Simple Salvo Model)

A Optimal Ratio σ_2^2/α^2.

σ_1^2/α^2	n					
	2	4	8	16	32	64
2	0.5	0.6	1.3	2.2	3.6	5.6
4	0.5	1.1	2.0	3.4	5.3	8.1
6	0.5	1.4	2.6	4.3	6.6	10.0
8	0.7	1.8	3.1	5.0	7.7	11.5
10	0.9	2.1	3.6	5.7	8.7	12.9
12	1.0	2.3	4.0	6.3	9.5	14.1

B Maximum Salvo Kill Probability

P_{max}

σ_1^2/α^2	n					
	2	4	8	16	32	64
2	0.450	0.622	0.780	0.904	0.973	0.996
4	0.293	0.431	0.599	0.768	0.899	0.971
6	0.217	0.334	0.489	0.665	0.826	0.935
8	0.173	0.274	0.416	0.588	0.761	0.896
10	0.144	0.233	0.363	0.528	0.707	0.858
12	0.124	0.204	0.323	0.481	0.659	0.823

Simple Salvo Model, we recall the approximate formula (3.22) given in Chapter III.

$$\sigma_2^2 = [\frac{1}{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2\{p^*(x,y) - D(x,y)\}dx\,dy\,,0]^+ . \qquad (6.20)$$

In the case of the Cookie Cutter Salvo Model,

$$\frac{1}{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2\, D(x,y)\, dx\,dy = \alpha^2/2 ,$$

whereas in the Simple Salvo Model,

$$\frac{1}{A}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2\, D(x,y)\, dx\,dy = \alpha^2 .$$

Therefore, (6.20) states that the optimal ballistic dispersion of the Cookie Cutter Salvo Model is larger than that of the Simple Salvo Model approximately by $\alpha^2/2$:

$$\sigma_{2c}^2/\alpha^2 - \sigma_2^2/\alpha^2 = 0.5 . \qquad (6.21)$$

Now we subtract entities in Table 16A from those in Table 15A and get the difference $\sigma_{2c}^2/\alpha^2 - \sigma_2^2/\alpha^2$ which is tabulated in Table 17A. The values are positive with only one exception, and the predicted value 0.5 is observed in the upper right

Table 17 Comparison of Cookie Cutter Salvo Model and Simple Salvo Model

A Difference of Optimal Ratio $\sigma_{2c}^2/\alpha^2 - \sigma_2^2/\alpha^2$.

	n					
σ_1^2/α^2	2	4	8	16	32	64
2	-0.1	0.4	0.4	0.5	0.5	0.5
4	0.3	0.5	0.6	0.5	0.5	0.5
6	0.7	0.7	0.6	0.6	0.5	0.5
8	0.9	0.7	0.7	0.6	0.6	0.5
10	0.9	0.8	0.8	0.7	0.6	0.5
12	1.1	1.0	0.9	0.7	0.6	0.6

B Difference of Maximum P's

$P_{cmax} - P_{max}$

	n					
σ_1^2/α^2	2	4	8	16	32	64
2	0.026	0.004	0.002	0.001	0.000	0.000
4	0.007	0.004	0.002	0.001	0.000	0.000
6	0.007	0.005	0.003	0.002	0.000	0.001
8	0.009	0.007	0.004	0.002	0.001	0.001
10	0.010	0.008	0.005	0.003	0.001	0.001
12	0.011	0.009	0.006	0.003	0.002	0.001

corner of the table. In the below left corner of the table, the figures are not close to the value 0.5, but these cases correspond to low salvo kill probability.

In Table 17B, the difference of the maximum probabilities of each model is tabulated. The figures are all positive, and lead to the speculation that $P_{c\,max} > P_{max}$ in all cases despite the fact that P_c may be smaller than P when σ_2 is not chosen optimally. The author has not yet succeeded in obtaining a proof of this speculation, but it might be the counterpart for $n > 1$ of Proposition 6.1 for $n = 1$.

In Chapter V, we developed an approximate method of computing the salvo kill probability which can be applied to the Cookie Cutter Salvo Model. The salvo kill probability is given by

$$P = 1 - K(n,0) , \tag{5.25}$$

where

$$K(n,j) = \int_0^\infty \{1 - (\mu + \beta t/\rho) e^{-t/\rho}\}^n e^{-(j/\rho+1)t}\, dt \tag{5.27}$$

satisfies the recurrence relation,

$$K(n,j) = \frac{n}{n+j+\rho} K(n-1,j) + \frac{\rho}{n+j+\rho}(1-\mu)^n - \frac{n\beta}{n+j+\rho} K(n-1,j+1) , \tag{5.28}$$

$$n = 1,2, \dots ; \quad j = 0,1, \dots$$

with

$$K(0,j) = \frac{\rho}{j+\rho}\ , \quad j = 1,2,\ \ldots \tag{5.29}$$

The parameters are given by (5.22), (5.23), and (5.24).

$$\mu = (\alpha^2/S^2)(1 - \alpha^2/2\sqrt{3}\ S^2)\ ,$$

$$\beta = \alpha^4/2\sqrt{3}\ S^4\ , \tag{6.22}$$

$$S^2 = \sigma_2^2 + \alpha^2(1 - 1/\sqrt{3})/2\ .$$

Using these formulae together with the golden section search calculation, we obtained Table 18. Table 18A gives the approximate optimal ballistic dispersion of the Cookie Cutter Salvo Model, and the associated salvo kill probability is in Table 18B.

In Table 19A, the difference between the optimal σ_{2c}^2/α^2 given in Table 15A and the approximate σ_{2A}^2/α^2 in Table 18A is tabulated. The difference decreases towards the upper right corner of the table.

In Table 19B, the salvo kill probability of the Cookie Cutter Salvo Model associated with the approximately optimal ballistic dispersion mentioned above is given. It corresponds to the maximum P given in Table 15B. In spite of the discrepancies observed in Table 19A, the approximate σ_{2A}^2/α^2 gives almost the same kill probability as the true σ_{2c}^2/α^2 gives.

Table 18 Optimal Ballistic Dispersion and the Associated Salvo Kill Probability

(Cookie Cutter Salvo Model, Approximation)

A Optimal Ratio σ_{2A}^2/α^2.

σ_1^2/α^2	n = 2	4	8	16	32	64
2	0.5	1.0	1.7	2.7	4.1	6.1
4	0.7	1.5	2.5	3.9	5.8	8.6
6	1.0	1.9	3.1	4.8	7.1	10.5
8	1.2	2.2	3.6	5.5	8.2	12.0
10	1.3	2.5	4.1	6.2	9.2	13.4
12	1.5	2.8	4.5	6.8	10.0	14.7

B Maximum Salvo Kill Probability

P_A

σ_1^2/α^2	n = 2	4	8	16	32	64
2	0.475	0.625	0.782	0.905	0.973	0.996
4	0.297	0.432	0.600	0.768	0.899	0.971
6	0.218	0.334	0.490	0.665	0.826	0.935
8	0.173	0.274	0.416	0.588	0.762	0.896
10	0.144	0.234	0.363	0.528	0.707	0.358
12	0.124	0.204	0.323	0.481	0.659	0.323

Table 19 Accuracy of the Approximation

(Cookie Cutter Salvo Model)

A Error in the Optimal Ratio, $\sigma_{2C}^2/\alpha^2 - \sigma_{2A}^2/\alpha^2$.

	n					
σ_1^2/α^2	2	4	8	16	32	64
2	-0.1	0.0	0.0	0.0	0.0	0.0
4	0.1	0.1	0.1	0.0	0.0	0.0
6	0.2	0.2	0.1	0.1	0.0	0.0
8	0.4	0.3	0.2	0.1	0.1	0.0
10	0.5	0.4	0.3	0.2	0.1	0.0
12	0.6	0.5	0.4	0.2	0.1	0.0

B Salvo Kill Probability Associated with the Approximately Optimal Ballistic Dispersion.

$P(\sigma_{2A}^2)$

	n					
σ_1^2/α^2	2	4	8	16	32	64
2	0.475	0.626	0.782	0.905	0.973	0.996
4	0.300	0.435	0.601	0.769	0.899	0.971
6	0.223	0.339	0.492	0.667	0.826	0.935
8	0.181	0.280	0.420	0.590	0.762	0.897
10	0.153	0.241	0.368	0.531	0.708	0.859
12	0.134	0.212	0.328	0.484	0.661	0.823

VII. SALVO MODELS WITHOUT CIRCULAR SYMMETRY

The approximate formula for the optimal ballistic dispersion presented in Chapter III covers the salvo models without circular symmetry. In this chapter the formulae for computing the salvo kill probability of these models are dealt with.

Suppose that a salvo of n weapons is fired against a target. The impact point error is composed of two parts, one being common to all the weapons, and another being round-to-round dispersion. The random target position (U,V) with respect to the coordinate system (x,y) is assumed to be elliptical normal centered at (0,0) with variances σ_u^2 and σ_v^2 respectively.

$$f_1(u,v) = \frac{1}{2\pi\,\sigma_u\sigma_v}\,e^{-\frac{u^2}{2\sigma_u^2}-\frac{v^2}{2\sigma_v^2}} . \qquad (7.1)$$

The impact points of the ith weapon (X_i,Y_i) are independent and identically distributed elliptical normal random variables centered at (0,0) with variances σ_x^2 and σ_y^2, respectively.

$$f_2(x,y) = \frac{1}{2\pi\,\sigma_x\sigma_y}\,e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}} . \qquad (7.2)$$

The kill probability D by a single weapon conditional on the target being at (u,v) and the weapon impact point being (x,y) is a function of u-x and v-y only, and is given by the elliptical Gaussian function with parameters α_x and α_y. Specifically,

$$D(u-x,v-y) = e^{-\frac{(u-x)^2}{2\alpha_x^2} - \frac{(v-y)^2}{2\alpha_y^2}} . \qquad (7.3)$$

It might be an appropriate model, e.g., for a ground target vs. a weapon impacting the ground obliquely, because in this case the scattering of the fragments and thus the conditional kill probability D is by no means circular symmetric as assumed in the previous chapters. For brevity, we call this model the Elliptical Normal Salvo Model.

The kill probability of a single weapon conditional on (U,V) being (u,v) is given by

$$p(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(u-x,v-y) f_2(x,y)\, dx\, dy ,$$

and using (7.2) and (7.3) we get

$$p(u,v) = \frac{\alpha_x \alpha_y}{\sqrt{(\alpha_x^2+\sigma_x^2)(\alpha_y^2+\sigma_y^2)}} e^{-\frac{u^2}{2(\alpha_x^2+\sigma_x^2)} - \frac{v^2}{2(\alpha_y^2+\sigma_y^2)}} \qquad (7.4)$$

The salvo kill probability is then given by

$$P = 1 - \frac{1}{2\pi\,\sigma_u\sigma_v}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{1-p(u,v)\}^n\, e^{-\frac{u^2}{2\sigma_u^2}-\frac{v^2}{2\sigma_v^2}}\,dudv\ . \qquad (7.5)$$

From this equation, Grubbs derived the following formula:

Proposition 7.1 (Grubbs)

The kill probability of the Elliptical Normal Salvo Model stated above is given by

$$P = \sum_{j=1}^{N}(-1)^{j-1}\binom{n}{j}\mu^j\sqrt{\frac{\rho_x\rho_y}{(j+\rho_x)(j+\rho_y)}}\ , \qquad (7.6)$$

where

$$\mu = \alpha_x\alpha_y/\sqrt{(\alpha_x^2+\sigma_x^2)(\alpha_y^2+\sigma_y^2)}\ ,$$

$$(7.7)$$

$$\rho_x = (\alpha_x^2+\sigma_x^2)/\sigma_u^2\ , \quad \rho_y = (\alpha_y^2+\sigma_y^2)/\sigma_v^2\ .$$

The Grubbs formula is not suited for calculation when n is large, since it is an alternating series. To overcome this difficulty, Breaux and Mohler [Ref. 13] gave a method for calculating the kill probability based on an expansion of $(1-z)^n$ in Jacobi polynomials rather than as a binomial series. The series is found to converge to the true value with less than n terms, which is very attractive for the calculation of salvo kill probability with a large n.

The author has tried to find alternative ways of calculating the salvo kill probability which are effective even for the cases with large n's, but at present, the obtained results are not promising. The following propositions are given not as efficient algorithms for calculating the salvo kill probability, but as possible hints for developing more efficient ways of calculation.

For the Elliptical Normal Salvo Model, we have a proposition similar to Proposition 2.5. We omit the proof, since it closely parallels that of Proposition 2.5.

Proposition 7.2

The salvo miss probability Q(n) of the Elliptical Normal Salvo Model characterized by (7.1), (7.2), and (7.3) with the weapon number n,

$$Q(n) = \frac{\sqrt{\rho_x \rho_y}}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \mu e^{-s^2-t^2})^n e^{-\rho_x s^2 - \rho_y t^2} \, ds \, dt \qquad (7.8)$$

satisfies the relationship

$$Q(n) = \sum_{k=0}^{n} \binom{n}{k} (1-\mu)^{n-k} \mu^k q(k) , \qquad (7.9)$$

where

$$q(k) = \frac{\sqrt{\rho_x \rho_y}}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - e^{-s^2-t^2})^n e^{-\rho_x s^2 - \rho_y t^2} \, ds \, dt . \qquad (7.10)$$

In the case of Simple Salvo Model, q(k) had a simple recurrence relationship, but we have not succeeded in finding such a relationship in this case.

Corresponding to the recurrence formula for the salvo kill probability presented in Chapter II, we have the next propositions: The series given in Proposition 7.3 is an infinite series, but for a small δ/ρ it will converge rapidly.

Proposition 7.3

The salvo miss probability of the Elliptical Normal Salvo Model with n weapons is given by

$$Q(n) = \sqrt{1-\delta^2/\rho^2} \sum_{k=0}^{\infty} \frac{(2k)!}{2^k (k!)^2} (\delta/\rho)^{2k} J(n,2k) , \tag{7.11}$$

where

$$J(n,k) = \frac{1}{k!} \int_0^{\infty} (1-\mu e^{-t/\rho})^n e^{-t} t^k \, dt ,$$

with

$$\begin{aligned} \rho &= \frac{1}{2}\left(\frac{\alpha_x^2+\sigma_x^2}{\sigma_u^2} + \frac{\alpha_y^2+\sigma_y^2}{\sigma_v^2}\right) , \\ \delta &= \frac{1}{2}\left(\frac{\alpha_x^2+\sigma_x^2}{\sigma_u^2} - \frac{\alpha_y^2+\sigma_y^2}{\sigma_v^2}\right) , \end{aligned} \tag{7.12}$$

and

$$\mu = \frac{\alpha_x \alpha_y}{\sqrt{(\alpha_x^2 + \sigma_x^2)(\alpha_y^2 + \sigma_y^2)}} .$$

Proof

In (7.8), let $s = r\cos\theta$, $t = r\sin\theta$, and $(\rho_x+\rho_y)/2 = \rho$, $(\rho_x-\rho_y)/2 = \delta$. Then

$$\rho_x s^2 + \rho_y t^2 = r^2(\rho + \delta\cos 2\theta)$$

and therefore Q(n) is

$$Q(n) = \frac{\sqrt{\rho_x\rho_y}}{\pi} \int_0^\infty (1 - \mu e^{-r^2})^n e^{-\rho r^2} \{\int_0^{2\pi} e^{-\delta r^2 \cos 2\theta} d\theta\} r\,dr$$

$$= 2\sqrt{\rho_x\rho_y} \int_0^\infty (1 - \mu e^{-r^2})^n e^{-\rho r^2} I_0(\delta r^2)\; r\,dr \quad ,$$

where $I_0(z)$ is the modified Bessel function of the first kind with order 0, and has an expansion formula

$$I_0(z) = \sum_{k=0}^{\infty} (\frac{1}{k!})^2 (\frac{z}{2})^{2k} . \qquad (7.13)$$

Thus

$$Q(n) = \sqrt{\rho_x \rho_y}/\rho \int_0^\infty (1-\mu e^{-t/\rho})^n e^{-t} I_0(\delta t/\rho)\,dt$$

$$= \sqrt{\rho_x \rho_y} \sum_{k=0}^\infty \frac{(2k)!}{2^k (k!)^2} (\delta/\rho)^{2k} \frac{1}{(2k)!} \int_0^\infty (1-\mu e^{-t/\rho})^n e^{-t} t^{2k}\,dt.$$

□

Proposition 7.4

The integral J(n,k) which has two integer arguments and two positive parameters μ and ρ,

$$J(n,k) = \frac{1}{k!} \int_0^\infty (1-\mu e^{-t/\rho})^n e^{-t} t^k\,dt \qquad (7.14)$$

satisfies the following recurrence relation.

$$J(n,k) = \frac{n}{n+\rho} J(n-1,k) + \frac{\rho}{n+\rho} J(n,k-1) , \qquad (7.15)$$

$$n = 1,2, \ldots ; \quad k = 1,2, \ldots$$

$$J(n,0) = \frac{n}{n+\rho} J(n-1,0) + \frac{\rho}{n+\rho} (1-\mu)^n , \quad n = 1,2, \ldots \qquad (7.16)$$

$$J(0,k) = 1 , \quad k = 1,2, \ldots \qquad (7.17)$$

Proof

$$J(n,k) = \frac{1}{k!}\int_0^\infty (1-\mu e^{-t/\rho})^{n-1}\, e^{-t}\, t^k\, dt$$

$$- \mu \frac{1}{k!}\int_0^\infty (1-\mu e^{-t/\rho})^{n-1} e^{-t/\rho-t}\, t^k\, dt \ .$$

The first term in the right-hand-side is J(n-1,k). If the second term is integrated by parts, then

$$J(n,k) = J(n-1,k) - \frac{1}{k!}\frac{\rho}{n}(1-\mu e^{-t/\rho})^n e^{-t}\, t^k \Big|_0^\infty$$

$$+ \frac{1}{k!}\frac{\rho}{n}\int_0^\infty (1-\mu e^{-t/\rho})^n e^{-t}\, k\, t^{k-1}\, dt$$

$$- \frac{1}{k!}\frac{\rho}{n}\int_0^\infty (1-\mu e^{-t/\rho})^n e^{-t}\, t^k\, dt$$

Therefore, for $k \neq 0$, we have

$$J(n,k) = J(n-1,k) + \frac{\rho}{n} J(n,k-1) - \frac{\rho}{n} J(n,k) \ .$$

From this relation (7.15) is immediate. If $k = 0$,

$$J(n,0) = J(n-1,0) + \frac{\rho}{n}(1-\mu)^n - \frac{\rho}{n} J(n,0) \ ,$$

and therefore, (7.16) results. The relation (7.17) is obvious.

□

It is noted that all the terms in the expansion (7.11) are positive, and that the integral J(n,0) is the salvo miss probability of the Simple Salvo Model dealt with in Chapter II. Therefore we have

Corollary 7.5

A lower bound to the salvo miss probability of the Elliptic Normal Salvo Model characterized by (7.1), (7.2) and (7.3) with weapon number n is given by

$$Q_E(n) = \sqrt{1 - \delta^2/\rho^2}\; Q(n) \;, \qquad (7.18)$$

where Q(n) is the miss probability of the Simple Salvo Model by n weapons with

$$\rho = \frac{1}{2}\left(\frac{\alpha_x^2 + \sigma_x^2}{\sigma_u^2} + \frac{\alpha_y^2 + \sigma_y^2}{\sigma_v^2}\right) ,$$

$$\delta = \frac{1}{2}\left(\frac{\alpha_x^2 + \sigma_x^2}{\sigma_u^2} - \frac{\alpha_y^2 + \sigma_y^2}{\sigma_v^2}\right) ,$$

$$\mu = \frac{\alpha_x \alpha_y}{\sqrt{(\alpha_x^2 + \sigma_x^2)(\alpha_y^2 + \sigma_y^2)}} .$$

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